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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. V.

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LONDON MATHEMATICAL SOCIETY.

VOL. V.

TENTH SESSION, 1873–4.

November 13th, 1873.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

Prof. CAYLEY, F.R.S., in the Chair.

The Reports of the Treasurer and Secretaries were read and adopted. From the Report of the Secretaries, it appeared that the number of Members was the same (113) as at the last General Meeting, held November 14th, 1872.

The Obituary of the Society, during the Session, comprised the names of Mr. Archibald Smith, F.R.S., Mr. Jacob Waley, M.A., and Mr. G. O'Hanlon.

The communications made to the Society during the past Session had been as follows:

"Remarks on some Recent Generalisations of Algebra:" valedictory Presidential Address by Dr. W. Spottiswoode, F.R.S.

"On a Theorem relating to Polyhedra analogous to Mr. Cotterill's Theorem on Plane Polygons:" by Prof. W. K. Clifford.

"Investigation of the Disturbance produced by a Spherical Obstacle on the Waves of Sound:" by the Hon. J. W. Strutt.

"Sur l'Intégration des Fonctions Circulaires:" by M. Hermite.

"On the Mechanical Description of a Cubic Curve:" by Prof. Cayley.

"On the Mechanical Description of certain Quartic Curves by a Modified Oval Chuck:" by the same.

"On Geodesic Lines, especially those of a Hyperboloid:" by the same.

"On a Deduction from Von Staudt's Property of Bernoulli's Numbers:" by Mr. J. W. L. Glaisher.

"Geometry on an Ellipsoid:" by Prof. Clifford.

"On Parallel Surfaces:" by Mr. S. Roberts.

"On the Greatest Common Divisors of the Minor Determinants of a Rectangular Matrix of which the Constituents are Integral Numbers:" by Prof. H. J. S. Smith.

"On an Arithmetical Demonstration of a Theorem in the Integral Calculus:" by the same.

"Summation of certain Series:" by Prof. Wolstenholme.

"On the Higher Singularities of Plane Curves;" and "On Systems of Linear Congruences:" by Prof. H. J. S. Smith.

"On an Extension of the term 'Area' to any Closed Circuit in Space:" by Mr. R. B. Hayward.

"On the Evaluation of a Class of Definite Integrals involving Circular Functions in the Numerator and Powers of the Variable only in the Denominator:" by Mr. J. W. L. Glaisher.

"Note on Normals and the Surface of Centres of an Algebraical Surface:" by Mr. S. Roberts.

"On the Calculation of the Value of the Theoretical Unit-Angle to a Great Number of Decimal Places:" by Mr. J. W. L. Glaisher.

"On Systems of Porismatic Equations, Algebraical and Trigonometrical:" by Prof. Wolstenholme.

"Note on Epicycloids and Hypocycloids:" by the same.

"Locus of Point of Concourse of Perpendicular Tangents to a Cardioid:" by the same.

"Elliptic Motion under Acceleration Constant in Direction:" by the same.

"Locus of Point of Concourse of Tangents to an Epicycloid, inclined to each other at a Constant Angle:" by the same.

"On the Theory of a System of Electrified Conductors:" by Prof. J. Clerk Maxwell.

"On the Focal Lines of a Refracted Pencil:" by the same.

"On an Application of the Theory of Unicursal Curves:" by M. Hermite.

"On a Rational Quintic Correspondence of Two Points in a Plane:" by Prof. Cayley.

"Bicursal Curves:" by the same.

"Plan of a Curve-Tracing Apparatus:" by the same.

"Note on the Plückerian Characteristics of Epi- and Hypo-trochoids, &c.:" by Mr. S. Roberts.

"Some General Theorems relating to Vibrations:" by Hon. J. W. Strutt.

"Invariant Conditions of Multiple-Concurrence of Two Conics:" by Mr. J. J. Walker.

"On a New Form of Biquaternion, being the Ratio of Two Systems of Forces:" by Prof. W. K. Clifford.

"Further Note on Geodesic Lines:" by Prof. Cayley.

Shorter communications were also made by Prof. Henrici, Mr. J. J. Walker, Hon. J. W. Strutt, Mr. M. Jenkins, Mr. J. Macleod, and Prof. W. K. Clifford.

The same Mathematical Journals had been subscribed for as in the preceding Session.

The Proceedings had been sent to the Mathematical Society of Paris, and to the University Library, Cambridge, in addition to the persons and Societies named in the last Report.

The Society had received the Bulletin of the Mathematical Society of Paris, in addition to those mentioned in the last Report.

The following presents had been made to the Society:—

"Ueber Systeme von Functionen mehrer Variabeln," von L. Kronecker, 2 parts (extracted from "Monatsbericht"): from the Author.

"On the Law of Facility of Errors of Observations;" and "On the Method of Least Squares," by J. W. L. Glaisher: from the Author.

"The Theory of Screws, a Geometrical Study of the Kinematics, Equilibrium, and Small Oscillations of a Rigid Body," by R. Stawell Ball: from the Author.

"Nautical Almanac for 1876," presented by authority of Lords Commissioners of the Admiralty.

"Vergleichende Betrachtungen über neuere geometrische Forschungen," von Dr. Felix Klein: from the Author.

"Royal Society's Catalogue of Scientific Papers," vol. vi.

"Catalogue of the Mathematical and Scientific Library of the late Charles Babbage."

"A Catalogue of a Collection of Models of Ruled Surfaces, constructed by M. Fabre de Lagrange, with an Appendix containing an account of the application of analysis to their investigation and classification," by C. W. Merrifield: from the Author.

"Annual of the Royal School of Naval Architecture, No. 3:" from Mr. Merrifield.

"On the Contact of Surfaces; and on some recent Generalisations of Algebra:" by W. Spottiswoode, F.R.S.: from the Author.

"A Complete Practical Treatise on the Nature and Use of Logarithms, and on Plane Trigonometry; with Logarithmic and Trigonometrical Tables;" by James Elliot (with Key): from the publishers.

"Salmon's Higher Plane Curves:" from the Author.

"Electricity and Magnetism" (2 vols.), by Prof. J. Clerk Maxwell: from the Author.

"Sur les trajectoires des points d'une droite mobile dans l'espace, et

démonstration géométrique d'une proposition due à M. Bertrand;" by M. Mannheim: from the Author.

"An Easy Rule for filling up all Magic Squares;" by S. M. Drach: from the Author.

Memoirs, Notes &c., by the late Prof. De Morgan: from Mrs. De Morgan.

"Untersuchungen über Elasticität unter Berücksichtigung der Wärme, und über das Ellipsoid von kleinsten Volumen bei gegebenem Flächeninhalt einer Anzahl von Centralschnitten, und über die Transformation der Electricitätsgleichungen in allgemeine orthogonale Coordinaten;" by C. W. Borchardt: from the Author.

"Sur une classe remarquable de courbes et de surfaces algébriques, et sur la théorie des imaginaires," par G. Darboux: from the Author.

Memoirs from det Kgl. Norske Universitet à Christiania.

After the Reports had been read, and Mr. H. Perigal, F.R.A.S., elected Auditor, the Meeting proceeded to the election of the new Council.

The following gentlemen were declared by the Scrutators (Messrs. W. Marsham Adams and R. Wormell) duly elected:—

President, Dr. Hirst, F.R.S.; Vice-Presidents, Prof. Cayley, F.R.S.; *J. J. Sylvester, Esq., LL.D., F.R.S., and W. Spottiswoode, Esq., LL.D., F.R.S.; Treasurer, Samuel Roberts, Esq., M.A.; Secretaries, M. Jenkins, Esq., M.A., and R. Tucker, Esq., M.A.; other Members of the Council, Prof. Clifford, M.A., Thomas Cotterill, Esq., M.A., J. W. L. Glaisher, Esq., B.A., F.R.A.S., Rev. R. Harley, F.R.S., Prof. Henrici, Dr.Phil., C. W. Merrifield, Esq., F.R.S., *The Right Hon. Lord Rayleigh, F.R.S., Prof. H. J. S. Smith, F.R.S., and J. J. Walker, Esq., M.A. (The gentlemen marked * were not on the Council for the Session 1872-73.)

Mr. G. S. Carr was elected a Member of the Society. Mr. R. R. Webb, B.A., Fellow of St. John's College, and Lecturer at Emmanuel College, Cambridge, Mr. Horace Lamb, B.A., Fellow of Trinity College, Cambridge, and the Rev. M. M. U. Wilkinson, M.A., late Fellow of Trinity College, Cambridge, were proposed for election.

The Chairman laid before the Meeting a printed letter received from the University of Christiania, informing the Society of the death of Prof. Hansteen, under whose superintendence the Trigonometrical Survey of Norway was made.

Mr. Sylvester then gave a description of a new instrument for converting circular into general rectilinear motion, and into motion in conics and higher plane curves, and was warmly applauded at the close of his address.

Mr. Merrifield and the Chairman asked several questions upon the subject of the communication.

Several instruments were submitted for inspection.

Prof. Cayley having to leave the Meeting, requested Mr. Sylvester to take his place.

Mr. W. Marsham Adams exhibited his Mensurator and Cælometer, showing their mode of operation, and gave a short account of the objects to which they could be applied.

Mr. S. Roberts read a short Note on the expression of the Arc of a Cartesian by Elliptic Functions.

The following presents were received:—

"Rendiconti, serie ii. vol. v. fasc. viii.—xvi. (adunanza del Aprile 1872 to 1° & 7° Aug. 1872)."

"Memorie del Reale Istituto Lombardo, vol. xii. to iii. della serie iii. fasc. v."

"Rappresentazione piana di alcune superficie algebriche dotate di curve cuspidali, nota del Prof. Luigi Cremona," Bologna, 1872, through Prof. Cremona.

"Annali di Matematica," serie ii^{a}. tom. v. fasc. 4°., May, 1873.

"Monatsbericht," Feb. 1873 (2 parts); Mai (2 parts), 1873.

"Bulletin des Sciences Mathématiques et Astronomiques," July to November, 1873; and "Table des Matières," tome iv., 1er semestre, 1873.

"Journal of London Institution," No. 21.

"Transactions of the Royal Irish Academy," vol. xxiv. pts. xvi., xvii.; vol. xxv., pts. i.—iv.

"Proceedings of the Royal Irish Academy," vol. x. pt. iv. (1870); vol. i., series ii., Nos. 2 ('70-'71) to 8 ('72-'73): from the Academy.

"Appendixes to the volumes of Astronomical and Meteorological Observations made at the United States Naval Observatory during 1870 and 1871;" from Rear-Admiral B. F. Sands, Superintendent.

"Crelle," 76 Band, zweites und drittes Heft, 1873.

"Journal of the Institute of Actuaries," No. xcii. July, 1873, vol. xvii. pt. vi., and List of Members.

Carte de Visite of Lord Rayleigh, F.R.S., taken by himself.

Wilson's "Elementary Geometry," 3rd edition, 1873: from the Author.

"Proceedings of the Royal Society," Nos. 145, 146.

"Note sur la différentiation et l'intégration d'une intégrale multiple par rapport à une constante;" par D. Bierens de Haan. (Extrait des Archives Néerlandaises, t. vi., 1871.)

"La Méthode d'Euler pour l'intégration de quelques équations différentielles linéaires démontrée à l'aide de l'équation intégrante;" par D. B. de Haan. (The same, t. vii., 1872.)

"On the Right Ascensions of the Equatorial Fundamental Stars, and the Corrections necessary to reduce the Right Ascensions of Different Catalogues to a Mean Homogeneous System;" by Simon Newcomb (1872).

"Report on the Difference of Longitude between Washington and St. Louis," by William Harkness. Results of Washington Observations, 1853 to 1860. United States Naval Observatory, Washington, 1872.

"Catalogue of Stars observed at the United States Naval Observatory during the years 1845 to 1871;" Washington, 1873. From the Naval Observatory, Washington City.

"The Problem of Pythagoras;" by W. Marsham Adams, B.A.

"Handbooks to the Mensurator and Cælometer" (in duplicate); by the same: from the Author.

Note on the expression of the length of the arc of a Cartesian by Elliptic Functions. By Samuel Roberts, M.A.

Having observed that the periphery of a Cartesian oval can be expressed by means of elliptic quadrants, when the axial foci are real,* I inferred that the general expression for a Cartesian arc might possibly be reduced to elliptic functions, by some real or imaginary transformation.

I propose now to show that this is really the case.

Let the equation of the Cartesian be

$$\rho^2 - 2(a + b\cos\omega)\rho + c^2 = 0,$$

then the arc element is

$$ds = \frac{(a+b\cos\omega)\sqrt{(a^2+b^2-c^2+2ab\cos\omega)}\,d\omega}{\sqrt{(a+b\cos\omega)^2-c^2}} \pm \sqrt{(a^2+b^2-c^2+2ab\cos\omega)}\,d\omega;$$

and, if $2\theta = \omega$,

$$ds = \frac{2(a+b-2b\sin^2\theta)(1-\kappa''^2\sin^2\theta)\,d\theta}{\sqrt{(1-\kappa^2\sin^2\theta)(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)}} \pm 2\sqrt{(a+b)^2-c^2}\,\sqrt{(1-\kappa''^2\sin^2\theta)}\,d\theta \quad \ldots\ldots\ldots (a);$$

where κ^2, κ'^2, κ''^2 are written for

$$\frac{2b}{a+b+c}, \quad \frac{2b}{a+b-c}, \quad \frac{4ab}{(a+b)^2-c^2}.$$

The expression affected by the $\pm$ sign is already an elliptic arc-element.

As to the first part of the value of ds, I remark that $\kappa^2+\kappa'^2-\kappa^2\kappa'^2 = \kappa''^2$, and the expression in question is, in fact, a transformation of Jacobi's

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* Proceedings, May 12, 1870.

reducible form

$$\frac{\mathrm{F}(\sin^2\theta)\,d\theta}{\sqrt{(1-\xi^2\sin^2\theta)\,(1-\xi'^2\sin^2\theta)\,(1-\xi^2\xi'^2\sin^2\theta)}},$$

for a particular determination of F.

If we write Q for the radical in the denominator of the first part of (a), and k^2, k'^2 for $1-\kappa^2$, $1-\kappa'^2$, it is known that

$$\int_0^\theta \frac{d\theta}{\mathrm{Q}} = \frac{1}{2kk'}\left\{\mathrm{F}\left(\frac{\kappa\kappa'}{1+kk'},\ \theta_1\right) - \mathrm{F}\left(\frac{\kappa\kappa'}{1-kk'},\ \theta_2\right)\right\},$$

$$\int_0^\theta \frac{\sin^4\theta\,d\theta}{\mathrm{Q}} = \frac{1}{2kk'}\left\{\frac{1}{1+kk'}\mathrm{F}\left(\frac{\kappa\kappa'}{1+kk'},\ \theta_1\right) - \frac{1}{1-kk'}\mathrm{F}\left(\frac{\kappa\kappa'}{1-kk'},\ \theta_2\right)\right\} \quad \ldots\ldots\ldots\ldots (\mathrm{A}),$$

$$\int_0^\theta \frac{\sin^2\theta\,d\theta}{\mathrm{Q}} = \frac{1}{2\kappa^2\kappa'^2}\left\{\mathrm{E}\left(\frac{\kappa\kappa'}{1+kk'},\ \theta_1\right) + \mathrm{E}\left(\frac{\kappa\kappa'}{1-kk'},\ \theta_2\right)\right\}$$
$$+ \frac{1}{2\kappa^2\kappa'^2kk'}\left\{\frac{\kappa^2-\kappa'^2-1-kk'}{1+kk'}\mathrm{F}\left(\frac{\kappa\kappa'}{1+kk'},\ \theta_1\right)\right.$$
$$\left. - \frac{\kappa^2+\kappa'^2-1+kk'}{1-kk'}\mathrm{F}\left(\frac{\kappa\kappa'}{1-kk'},\ \theta_2\right)\right\};$$

with the conditions

$$\tan\theta_1 = \frac{(1+kk')\tan\theta}{1-kk'\tan^2\theta}, \quad \tan\theta_2 = \frac{(1-kk')\tan\theta}{1+kk'\tan^2\theta},$$

for which expressions see Mr. Michael Roberts's Tract on Elliptic and Hyper-elliptic functions, chap. vi., pp. 62—66. Moreover since, when

$\omega = 0$, $\qquad \rho = a+b \pm \sqrt{(a-b)^2-c^2}$,

and, when $\omega = \pi$, $\qquad \rho = a-b \pm \sqrt{(a-b)^2-c^2}$,

measured in an opposite direction, it is seen that, if A, B, C, D are the successive points of intersection with the axis,

$$\begin{aligned}
\mathrm{CD} &= 2\sqrt{(a+b)^2-c^2} = 2\mathrm{R},\\
\mathrm{AB} &= 2\sqrt{(a-b)^2-c^2} = 2\mathrm{R}',\\
\mathrm{BD} &= 2a+\mathrm{R}-\mathrm{R}' \quad = 2\mathrm{R}_1,\\
\mathrm{AC} &= 2a-\mathrm{R}+\mathrm{R}' \quad = 2\mathrm{R}'_1,\\
\mathrm{AD} &= 2a+\mathrm{R}+\mathrm{R}' \quad = 2\mathrm{R}_2,\\
\mathrm{BC} &= 2a-\mathrm{R}-\mathrm{R}' \quad = 2\mathrm{R}'_2.
\end{aligned}$$

If we now substitute their present values for κ^2, κ'^2, k, k' in (A), and make use of the results thus obtained in the integral of (a), we get

$$s = \frac{1}{\mathrm{R}+\mathrm{R}'}\left\{(b-a)\,\mathrm{R}-(a+b)\,\mathrm{R}'\right\}\mathrm{F}\left(\frac{2b}{\mathrm{R}+\mathrm{R}'},\ \theta_1\right)$$
$$+ \frac{1}{\mathrm{R}-\mathrm{R}'}\left\{(b-a)\,\mathrm{R}+(a+b)\,\mathrm{R}'\right\}\mathrm{F}\left(\frac{2b}{\mathrm{R}-\mathrm{R}'},\ \theta_2\right)$$
$$+ 2a\left\{\mathrm{E}\left(\frac{2b}{\mathrm{R}+\mathrm{R}'},\ \theta_1\right) + \mathrm{E}\left(\frac{2b}{\mathrm{R}-\mathrm{R}'},\ \theta_2\right)\right\} \pm 2\mathrm{RE}\left(\frac{2\sqrt{ab}}{\mathrm{R}},\ \theta\right),$$

with the conditions

$$\tan\theta_1 = \frac{(R+R')\tan\theta}{R-R'\tan^2\theta}, \quad \tan\theta_2 = \frac{(R-R')\tan\theta}{R+R'\tan^2\theta}.$$

And further observing that

$$\frac{(b-a)R-(a+b)R'}{R+R'} = -a\left(1-\frac{4b^2}{(R+R')^2}\right),$$

$$\frac{(b-a)R+(a+b)R'}{R-R'} = -a\left(1-\frac{4b^2}{(R-R')^2}\right),$$

and making use of the known formula,

$$2E(c_1,\phi_1)-(1-c_1^2)F(c_1,\phi_1) = 2(1+c_1)E(c,\phi)-2c_1\sin\phi_1 \ldots\ldots (B),$$

$$c = \frac{2\sqrt{c_1}}{1+c_1}, \quad \tan(\phi_1-\phi) = \sqrt{1-c^2}\tan\phi,$$

we get the form

$$2R_1E\left(\frac{\sqrt{R_1^2-R_1'^2}}{R_1},\ \theta'\right)+2R_2E\left(\frac{\sqrt{R_2-R_2'^2}}{R_2},\ \theta''\right) \pm 2RE\left(\frac{\sqrt{R^2-R'^2}}{R},\ \theta\right)$$
$$-(R-R')\sin\theta_1-(R+R')\sin\theta_2,$$

with the conditions $\tan(\theta_1-\theta') = \frac{R_1'}{R_1}\tan\theta'$,

$$\tan(\theta_2-\theta'') = \frac{R_2'}{R_2}\tan\theta''.$$

It must be remarked, however, that on account of the complication of the limits and discontinuity of sign, the above result only shows that the arc of the Cartesian can be expressed in terms of arcs of the ellipses constructed on the axial diameters as principal axes. In order to follow the effect of the conditions among the limits, it is best to revert to the original form. Thus for a semi-periphery, the limits of θ are 0 and $\frac{\pi}{2}$. But in accordance with the integrations effected in Mr. M. Roberts's Tract, p. 67, we have

$$\frac{\text{periphery}}{2} = \frac{2}{R+R'}\left\{(b-a)R-(a+b)R'\right\}F\left(\frac{2b}{R+R'},\ \frac{\pi}{2}\right)$$
$$+4aE\left(\frac{2b}{R+R'},\ \frac{\pi}{2}\right) \pm 2RE\left(\frac{2\sqrt{ab}}{R},\ \frac{\pi}{2}\right),$$

or by (B), $= 2R_1E\left(\frac{\sqrt{R^2-R_1'^2}}{R_1},\ \frac{\pi}{2}\right) \pm 2RE\left(\frac{\sqrt{R^2-R'^2}}{R},\ \frac{\pi}{2}\right)$

= sum or difference of the semi-peripheries of two ellipses on BD, AC, and CD, AB as principal diameters.

If we take the external focus for origin, and therefore $b>a$, $b-a>c$,

other transformations are necessary, since $\frac{2b}{a+b-c}$, &c., are >1. And again, if we take c^2 negative and the middle focus as the origin, the radical has imaginary factors. The reduction of these forms would be somewhat laborious. As to Cartesians with a pair of imaginary foci, there is still more difficulty.

December 11th, 1873.

Prof. CAYLEY, F.R.S., Vice-President, in the Chair.

Messrs. Horace Lamb, B.A., R. R. Webb, B.A., and the Rev. M. M. U. Wilkinson, M.A., were elected Members of the Society. Mr. C. J. Lambert, M.A., Assistant Professor of Mathematics at the Royal Naval College, Greenwich, and Mr. R. Forsyth Scott, of St. John's College, Cambridge, were proposed for election.

The Auditor (Mr. Perigal) certified that he had examined the Treasurer's accounts, and found them correct.

Prof. Clifford gave an account of his paper on "the Graphic Representation of the Harmonic Components of a Periodic Motion."

Prof. Cayley spoke on the subject of Steiner's Surface, and exhibited a diagram.

Mr. Roberts and Prof. Cayley made some further extempore communications.

Lord Rayleigh mentioned an application of a general theorem, contained in a paper read before the Society in June last, to the vibrations of membranes. The theorem is to the effect that an increase in the potential energy of a vibrating system is attended by a rise in pitch, extending to all the fundamental modes. If the system consist of a uniformly stretched membrane with a fixed edge, it follows that any contraction of the boundary must cause an elevation of pitch, because the new state of things may be conceived to differ from the old merely by the introduction of an additional constraint. Springs (without inertia) are supposed to urge the line of the proposed boundary towards its equilibrium position, and gradually to become stiffer. At each step the vibrations become more rapid, until they approach a limit, corresponding to infinite stiffness of the springs and absolute fixity of their points of application. It is not assumed that the part cut off from the membrane has the same density as the rest, or even any density at all.

If the membrane be uniform, of given density, and stretched with a given tension, the frequency of vibration is a function of the size and

form. When the form is invariable, the frequency varies as the linear dimension.

There are not many forms for which the problem can be solved. Circles and circular sectors, rectangles, and one or two varieties of triangles, almost exhaust the list. The result just obtained, however, allows a tolerable estimate to be made in a considerable number of cases. For example, the pitch of a regular polygon is intermediate between those of the inscribed and circumscribed circles. A sector of a circle containing 60° is flatter than an equilateral triangle, whose side is equal to the radius of the circle.

Closer limits may usually be obtained by means of a second theorem. When the area of the membrane is given, it is easy to see that any elongation or projecting corners tend to raise the pitch. Among rectangles of given area, the square gives the gravest tone. It can be proved that of all figures differing but little from a circle, the exact circle is the gravest; and it is, no doubt, generally true that any membrane not a circle is higher in pitch than the circle of equal area. In estimating, therefore, the lower limit to the pitch of a regular polygon, it would be advantageous to replace the circumscribed circle by that of equal area. In the case of the hexagon, for example, the ratio of the radius of the circle of equal area to that of the inscribed circle is 1·050, so that the mean of the two limits cannot differ from the truth by so much as $2\frac{1}{2}$ per cent.

The following table shows the relative frequency of the gravest tone in certain calculable cases, the *area* of the membrane, as well as the mechanical conditions, being supposed constant:—

circle	$2{\cdot}404\sqrt{\pi}$	$= 4{\cdot}261$
square	$\sqrt{2}\,.\,\pi$	$= 4{\cdot}443$
quadrant of a circle	$\sqrt{\pi}\,.\,\frac{5{\cdot}135}{2}$	$= 4{\cdot}551$
sector of a circle 60°	$\sqrt{\frac{\pi}{6}}\,.\,6{\cdot}379$	$= 4{\cdot}616$
rectangle, ratio of sides 3 : 2	$\pi\sqrt{\frac{13}{6}}$	$= 4{\cdot}624$
equilateral triangle	$2\pi\sqrt{\tan 30°}$	$= 4{\cdot}774$
semicircle	$\sqrt{\frac{\pi}{2}}\,.\,3{\cdot}832$	$= 4{\cdot}803$
rectangle 2 : 1 right-angled isosceles triangle	$\pi\sqrt{\frac{5}{2}}$	$= 4{\cdot}967$
rectangle 3 : 1	$\pi\sqrt{\frac{10}{3}}$	$= 5{\cdot}736$

For instance, if a square and circle have the same area, the former is the higher in the ratio 4·443 : 4·261.

The following presents were received :—

"Crelle," 76 Band, 4es Heft; 77 Band, 1es Heft.

"Ueber die verschiedenen Sturmschen Reihen und ihre gegenseitigen Beziehungen," von L. Kronecker (from "Monatsbericht" vom Feb. 1873): from Prof. Kronecker.

"Ueber Deformationen elastischer isotroper Körper durch mechanische an ihrer Oberfläche wirkende Kräfte," von C. W. Borchardt (from "Monatsbericht" vom Juli, 1873): from the Author.

"Monatsbericht," Juni, Juli, and August, 1873.

"Alfred Clebsch, Versuch einer Darlegung und Würdigung seiner wissenschaftlichen Leistungen, von einigen seiner Freunde:" Leipzig, 1873.

"Verhandlungen der physicalisch-medicinischer Societät zu Erlangen," 1865—1867: Erlangen, 1867.

The same, zweites Heft, Mai 1867 bis Mai 1870: Erlangen, 1870.

"Sitzungsberichte der physicalisch-medicinischer Societät zu Erlangen," drittes Heft, Mai 1870 bis Aug. 1871; viertes Heft, Nov. 1871 bis Aug. 1872; fünftes Heft, Nov. 1872 bis Aug. 1873.

"Bulletin des Sciences Mathématiques et Astronomiques," Dec. 1873.

"Nautical Almanac" for 1877, presented by authority of Lords Commissioners of the Admiralty.

"Special Report on Immigration, accompanying information for immigrants relative to the prices and rentals of land, the staple products, facilities of access to market, &c.; to which are appended Tables, &c." By Edward Young, Ph.D., Chief of the Bureau of Statistics, Washington, 1872: from the Author.

"Jahrbuch über die Fortschritte der Mathematik," von Carl Ohrtmann, Felix Müller, Albert Wangerin; dritter Band, Jahrgang 1871; in three Heften; Heft 1 and 2, 1873: from Dr. Ohrtmann.

"Geometric Maps exhibiting the method of delineating Curves through the intersections of Trigonometrical Lines:" from the Author, H. Perigal.

Graphic Representation of the Harmonic Components of a Periodic Motion. *By* Prof. W. K. Clifford, M.A.

[*Read December 11th*, 1873.]

Fourier's theorem asserts that any motion having the period P may be decomposed into simple harmonic motions having periods P, $\frac{1}{2}$P, $\frac{1}{3}$P, &c.; and assigns the amplitudes and phases of these motions by means of definite integrals. In fact, if $\phi(x+2\pi) = \phi(x)$ for all values of x,

then $$\phi(x) = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \ldots + b_m \cos mx + \ldots$$
$$+ a_1 \sin x + a_2 \sin 2x + \ldots + a_m \sin mx + \ldots,$$

where $$\pi b_m = \int_{-\pi}^{+\pi} \phi(a) \cos ma \, da,$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi(a) \sin ma \, da;$$

and this is made applicable to the general case of periodic motion by putting $\frac{x}{2\pi} = \frac{t}{\mathrm{P}}$, where t is the time elapsed since the era of reckoning.

The terms $b_m \cos mx + a_m \sin mx$ constitute a simple harmonic motion of period $\frac{\mathrm{P}}{m}$; the object of the present communication is to represent this motion by a graphical construction.

If a right circular cylinder be made to revolve uniformly about its axis, while a pencil in contact with its surface has a rectilinear motion parallel to the axis, the pencil will trace out upon the cylinder a curve representing its motion. In particular, if this motion is simple harmonic and of a period equal to that of the revolution of the cylinder, the curve traced out will be an ellipse. The amplitude of the motion will be $r \cot \theta$, where r is the radius of the cylinder and θ the inclination of its axis to the plane of the ellipse; the phase at epoch is determined by the orientation of the major axis. This ellipse may thus be regarded as a graphical representation of the simple harmonic motion.

Now let the pencil have any arbitrary motion whose period is P. For convenience let us suppose that the axis of the cylinder is vertical. Then, if the cylinder be made to turn once round in the time P, a curve C_1 will be traced on it, representing the arbitrary periodic motion. Next let the cylinder turn round twice in the period P; a curve C_2 will be traced on it. And generally when the cylinder turns round m times in the time P, a curve C_m will be traced on it, going m times round the cylinder. All these curves C will be closed curves, because the motion is periodic.

At this point I call to mind Mr. Hayward's extension of the meaning of "area," whereby it is made to have direction as well as magnitude. Any closed contour, not necessarily plane, being given, the area of its projection on a plane is found to be a maximum when the plane has a certain aspect. The magnitude of this maximum area, considered as having this particular aspect, is called the area of the contour; and the area of the projection on any other plane is proportional to the cosine of the angle which it makes with the maximum plane. If, therefore, we know the area of the projection of a contour on any three planes at right angles, we can find the area of the projection on any other plane.

Now I say that it is possible to draw on the cylinder an ellipse which shall have the same area in magnitude and direction as the curve C_1. For the projection of this curve on a plane perpendicular to the axis (*i.e.*, a horizontal plane) is merely the circular section of the cylinder, which is the same as the projection of any ellipse traced on it. If therefore we cut the cylinder by a plane parallel to the maximum plane of the contour C_1, the elliptic section E_1 will have the same area as that contour in magnitude and direction.

The contour C_2 goes twice round the cylinder; therefore its projection on a plane perpendicular to the axis is the circle described twice in the same direction, and its area is twice that of the projection of any ellipse. If therefore we cut the cylinder by a plane parallel to the maximum plane of C_2, we shall obtain an ellipse E_2 whose area is half that of the contour C_2 and parallel to it. Similarly, the contour C_m goes m times round the cylinder, and a plane parallel to its maximum plane will determine an ellipse E_m whose area is $\frac{1}{m}$th of that of C_m.

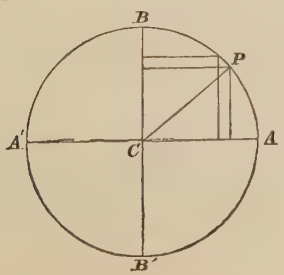
Now first let a circle be drawn on the cylinder whose height is the mean height of C_1. On this circle is the middle point of all the component oscillations.

Next, while the cylinder goes round once in the period P, let the pencil follow the ellipse E_1; it will then have a simple harmonic motion of period P, which is, in fact, the first or fundamental component. Then, while the cylinder goes round twice in the time P let the pencil follow the ellipse E_2: the resulting simple harmonic motion of period $\frac{P}{2}$ is the second component. Generally, while the cylinder goes round m times in the time P, let the pencil follow the ellipse E_m; this simple harmonic motion is the mth component.

The demonstration of this result is very simple. The values of a_m and b_m may be written as follows:

$$\pi b_m = \int_{-\pi}^{+\pi} \phi a \cos ma\, da = \frac{1}{m} \int_{a=-\pi}^{a=\pi} \phi a\, d \sin ma,$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi a \sin ma\, da = -\frac{1}{m} \int_{a-\pi}^{a=\pi} \phi a\, d \cos ma.$$



Suppose now that $\phi\alpha$ is set up vertically at P, when PCA $= m\alpha$, then $d\cos m\alpha$ is the element of CA, and $d \sin m\alpha$ is the element of CB; so that the differentials under the integral signs are respectively elements of the areas projected on vertical planes through AA′ and BB′. If in these integrals we write $b_m \cos\alpha + a_m \sin\alpha$ in place of $\phi\alpha$, we get the areas of the corresponding projections of the ellipse E_m; these are πb_m and πa_m respectively. Thus the area of C_m projected on three planes at right angles is m times that of the ellipse E_m; or the areas of the two curves have the same aspect and are in the ratio $m : 1$; which was to be proved.

On Steiner's Surface. *By* Prof. Cayley.

[*Read December* 11*th*, 1873.]

I have constructed a model and drawings of the symmetrical form of Steiner's Surface, viz., that wherein the four singular tangent planes form a regular tetrahedron, and consequently the three nodal lines (being the lines joining the mid-points of opposite edges) a system of rectangular axes at the centre of the tetrahedron. Before going into the analytical theory, I describe as follows the general form of the surface: take the tetrahedron, and inscribe in each face a circle (there will be, of course, two circles touching at the mid-point of each edge of the tetrahedron; each circle will contain, on its circumference at angular distances of 120°, three mid-points, and the lines joining these with the centre of the tetrahedron, produced beyond the centre, meet the opposite edges, and are in fact the before-mentioned lines joining the mid-points of opposite edges). Now truncate the tetrahedron by planes parallel to the faces so as to reduce the altitudes each to three-fourths of the original value, and from the centre of each new face round off symmetrically up to the adjacent three circles; and within each circle scoop down to the centre of the tetrahedron, the bounding surface of the excavation passing through the three right lines, and the sections (by planes parallel to the face) being in the neighbourhood of the face nearly circular, but as they approach the centre, assuming a trigonoidal form, and being close to the centre an indefinitely small equilateral triangle. We have thus the surface, consisting of four lobes united only by the lines through the mid-points of opposite edges, these lines being consequently nodal lines; the mid-points being pinch-points of the surface, and the faces singular planes, each touching the surface along the inscribed circle. The joining lines, produced indefinitely both ways, belong as nodal lines to the surface, but they are, outside

the tetrahedron, mere acnodal lines not traversed by any real sheet of the surface.

We may imagine the tetrahedron placed in two different positions, (1) resting with one of its faces on the horizontal plane, (2) with two opposite edges horizontal, or say with the horizontal plane passing through the centre of the tetrahedron and being parallel to two opposite edges; or what is the same thing, the nodal lines form a system of rectangular axes, one of them, say that of z, being vertical. And I proceed to consider, in the two cases respectively, the horizontal sections of the surface.

In the first case, the coordinates x, y, z, w may be taken to be the perpendicular distances of a point from the faces of the tetrahedron, w being the distance from the base. We have, if the altitude be h,*

$$x+y+z+w = h;$$

an equation which may be used to homogenize any equation not originally homogeneous; thus, for the plane $w=\lambda$, of altitude λ, we have

$$w = \frac{\lambda}{h}(x+y+z+w),$$

or, what is the same thing,

$$w = \frac{\lambda}{h-\lambda}(x+y+z).$$

The equation of the surface is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{w} = 0,$$

and if we herein consider w as having the last-mentioned value, the equation will belong to the section by the plane $w=\lambda$. I remark that the section of the tetrahedron, by this plane, is an equilateral triangle, the side of which is to an edge of the tetrahedron as $h-\lambda : h$. For a point in the plane of the triangle, if X, Y, Z are the perpendiculars on the sides, then

$$X+Y+Z = P,$$

(if for a moment P is the perpendicular from a vertex on the opposite side of the triangle, viz., we have $P = \frac{h-\lambda}{h}.p$, if p be the perpendicular for a face of the tetrahedron). And it is clear that x, y, z are propor-

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* I take the opportunity of remarking that in a regular tetrahedron, if s be the length of an edge, p the perpendicular from a summit on an edge (or altitude of a face), h the perpendicular from a summit on a face (or altitude of the tetrahedron), and q the distance between the mid-points of opposite edges, then

$$s = \frac{\sqrt{3}}{\sqrt{2}}h, \quad p = \frac{3}{2\sqrt{2}}h, \quad q = \frac{\sqrt{3}}{2}h.$$

The tetrahedron can be, by means of planes through the mid-points of the edges, at right angles thereto, divided into four hexahedral figures (8 summits, 6 faces, 12 edges, each face a quadrilateral); viz., in each such figure there are, meeting in a summit of the tetrahedron, three edges each $=\frac{1}{2}s$; meeting in the centre three edges, each $=\frac{1}{4}h$; and six other edges each $=\frac{1}{8}p$.

tional to X, Y, Z; we consequently have, for the equation of the section,

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{\frac{\lambda}{h-\lambda}}\,(X+Y+Z)=0,$$

where the coordinates X, Y, Z are the perpendicular distances from the sides of the triangle which is the section of the tetrahedron. To simplify, I write $\frac{\lambda}{h-\lambda}=2q+1$, that is $q=\frac{2\lambda-}{2h-2\lambda}$

the equation is

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{(2q+1)(X+Y+Z)}=0;$$

or, proceeding to rationalize, we have first

$$q\,(X+Y+Z)=\sqrt{YZ}+\sqrt{ZX}+\sqrt{XY},$$

and thence

$$q^2\,(X+Y+Z)^2-(YZ+YX+XY)=2\sqrt{XYZ}\,(\sqrt{X}+\sqrt{Y}+\sqrt{Z});$$

and finally

$$\{q^2\,(X+Y+Z)^2-YZ-ZX-XY\}^2=4\,(2q+1)\,XYZ\,(X+Y+Z),$$

viz., this is a quartic curve, having for double tangents the four lines $X=0$, $Y=0$, $Z=0$, $X+Y+Z=0$, the last of these being the line infinity touching the curve in two imaginary points, since obviously the whole real curve lies within the triangle. This is as it should be, the double tangents are the intersections of the plane $w=\lambda$ by the singular planes of the surface.

To find the points of contact, writing for instance $Z=0$, the equation becomes $q^2\,(X+Y)^2-XY=0$, that is $X^2+\left(2-\frac{1}{q^2}\right)XY+Y^2=0$;

whence

$$X=\left(-1+\frac{1}{2q^2}\pm\sqrt{\frac{1}{4q^4}-\frac{1}{q^2}}\right)Z,$$

giving the two points of contact equi-distant from the centre; these are imaginary if $q>\frac{1}{2}$, but otherwise real, which agrees with what follows. (See Table afterwards referred to.)

The nodal lines of the surface are $(x-y=0,\ z-w=0)$, $(y-z=0,\ x-w=0)$, $(z-x=0,\ y-w=0)$. Considering the first of these, we have for its intersection with the plane $w=\lambda$,

$$X=Y,\quad Z=\frac{\lambda}{h-\lambda}(X+Y+Z),\ =(2q+1)\,(X+Y+Z),$$

and the last equation gives

$$Z=(2q+1)\,(2X+Z),\quad \text{that is,}\quad 0=(2q+1)\,X+qZ,$$

so that for the point in question we have $X:Y:Z=-q:-q:2q+1$; and taking the perpendicular from the vertex on a side as unity, the values $-q$, $-q$, $2q+1$ will be absolute magnitudes. We thus see that the curve must have the three nodes $(2q+1,-q,-q)$, $(-q,2q+1,-q)$, $(-q,-q,2q+1)$; and it is easy to verify that this is so.

The curve will pass through the centre $X=Y=Z$, if

$$(9q^2-3)^2 = 12\,(2q+1),$$

that is,
$$3\,(3q^2-1)^2-4\,(2q+1) = 0,$$

or
$$(3q+1)^2\,(q-1) = 0.$$

If $q=1$, that is, $\lambda = 3\,(h-\lambda)$ or $\lambda = \frac{3}{4}h$, the equation is

$$(X^2+Y^2+Z^2+YZ+ZX+XY)^2-12XYZ\,(X+Y+Z) = 0,$$

where the curve is, in fact, a pair of imaginary conics meeting in the four real points $(3, -1, -1)$, $(-1, 3, -1)$, $(-1, -1, 3)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To verify this, observe that, writing

$$A = (Y-Z)\,(2X+Y+Z),$$
$$B = (Z-X)\,(X+2Y+Z),$$
$$C = (X-Y)\,(X+Y+2Z),$$

and therefore
$$A+B+C = 0,$$

the function in (X, Y, Z) is $= \frac{1}{2}(A^2+B^2+C^2)$, and thus the equation may be written in the equivalent forms

$$B^2+BC+C^2 = 0, \quad C^2+CA+A^2 = 0, \quad A^2+AB+B^2 = 0,$$

each of which shows that the curve breaks up into two imaginary conics. The foregoing value $q=1$, or $\lambda = \frac{3}{4}h$, belongs to the summit or highest real point of the surface.

In the case $3q+1 = 0$, that is,

$$q = -\frac{1}{3} = \frac{2\lambda-h}{2h-2\lambda}, \quad \text{or} \quad \lambda = \tfrac{1}{4}h,$$

the equation is

$$\{(X+Y+Z)^2-9\,(YZ+ZX+XY)\}^2 = 108\,XYZ\,(X+Y+Z),$$

which is, in fact, the equation of a curve having the centre or point $X=Y=Z$ for a triple point.

To verify this, write
$$X = \beta-\gamma+u,$$
$$Y = \gamma-\alpha+u,$$
$$Z = \alpha-\beta+u;$$

also
$$2\Delta = (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2,$$
$$\Omega = (\beta-\gamma)\,(\gamma-\alpha)\,(\alpha-\beta);$$

then we have
$$X+Y+Z = 3u,$$
$$YZ+ZX+XY = 3u^2-\Delta,$$
$$XYZ = u^3-u\Delta+\Omega,$$

and the equation is

$$\{9u^2-9\,(3u^2-\Delta)\}^2-324u\,(u^3-u\Delta+\Omega) = 0,$$

that is,
$$(-2u^2+\Delta)^2-4u\,(u^3-u\Delta+\Omega) = 0,$$

or finally
$$\Delta^2-4u\Omega = 0,$$

where the lowest terms in $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ are of the order 3, and the theorem is thus proved. The case in question, $q = -\frac{1}{3}$ or $\lambda = \frac{1}{4}h$, is where the plane passes through the centre of the tetrahedron.

$q = \frac{1}{2}, = \frac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{2}{3}h$; the equation is

$$(X^2+Y^2+Z^2-2YZ-2ZX-2XY)^2 = 128XYZ(X+Y+Z) = 0.$$

Here each of the lines $X=0$, $Y=0$, $Z=0$ is an osculating tangent having with the curve a 4-pointic intersection.

$q = 0, = \frac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{1}{2}h$; the equation is

$$(YZ+ZX+XY)^2-4XYZ(X+Y+Z) = 0,$$

that is, $$Y^2Z^2+Z^2X^2+X^2Y^2-2XYZ(X+Y+Z) = 0;$$

viz., each angle of the triangle is here a cusp.

$q = -\frac{1}{2}$, or $\lambda = 0$; the curve is here

$$\{X^2+Y^2+Z^2-2(YZ+ZX+XY)\}^2 = 0,$$

viz., the plane is here the base of the tetrahedron, and the section is the inscribed circle taken twice.

For tracing the curves, it is convenient to find the intersections with the lines $Y-Z=0$, $Z-X=0$, $X-Y=0$ drawn from the centre of the triangle to the vertices; each of these lines passes through a node, and therefore besides meets the curve in two points. Writing, for instance, $Y = X$, the equation becomes

$$\{q^2(2X+Z)^2-2XZ-X^2\}^2-4(2q+1)X^2Z(2X+Z) = 0;$$

viz., this is

$$\{qZ+(2q+1)X\}^2\{q^2Z^2+(4q^2-2q-4)XZ+(4q^2-4q+1)X^2\} = 0,$$

where the first factor gives the node. Equating to zero the second factor, we have

$$\left\{qZ+\left(2q-1-\frac{2}{q}\right)X\right\}^2 = X^2\left\{\left(2q-1-\frac{2}{q}\right)^2-4q^2+4q-1\right\}$$

$$= X^2.\frac{4}{q^2}(1-q)(1+2q);$$

or finally $$qZ = \left\{-2q+1+\frac{2}{q} \pm \frac{2}{q}\sqrt{(1-q)(1+2q)}\right\}X,$$

giving two real values for all values of q from $q=1$ to $q=-\frac{1}{2}$. See the Table afterwards referred to.

We may recapitulate as follows:

$q > 1$, or $\lambda > \frac{3}{4}h$; the curve is imaginary, but with three real acnodes, answering to the acnodal parts of the nodal lines.

$q = 1$, or $\lambda = \frac{3}{4}h$; the summit appears as a fourth acnode.

$q < 1 > \frac{1}{2}$, or $\lambda < \frac{3}{4}h > \frac{2}{3}h$; the curve consists of three acnodes and a trigonoid lying within the triangle and having the sides of the triangle for bitangents of imaginary contact.

$q = \frac{1}{2}$, or $\lambda = \frac{2}{3}h$; the curve consists of three acnodes and a trigonoid having the sides of the triangle for osculating tangents.

$q < \frac{1}{2} > 0$, or $\lambda < \frac{2}{3}h > \frac{1}{2}h$; the curve consists of three conjugate points and an indented trigonoid having the sides of the triangle for bitangents of real contact.

$q = 0$, or $\lambda = \frac{1}{2}h$; curve has the summits of the triangle for cusps.

$q < 0 > -\frac{1}{2}$, or $\lambda < \frac{1}{2}h > \frac{1}{4}h$; curve has three crunodes, or say it is a cis-centric trifolium.

$q = -\frac{1}{3}$, or $\lambda = \frac{1}{4}h$; curve has a triple point, or say it is a centric trifolium.

$q < -\frac{1}{3} > -\frac{1}{2}$, or $\lambda < \frac{1}{4}h > 0$; curve has three crunodes, or say it is a trans-centric trifolium.

$q = -\frac{1}{2}$, or $\lambda = 0$; curve is a two-fold circle.

$q < -\frac{1}{2}$, or $\lambda < 0$; the curve becomes again imaginary, consisting of three acnodes answering to the acnodal parts of the nodal lines.

For the better delineation of the series of curves, I calculated the following Table, wherein the first column gives a series of values of $\lambda : h$; the second the corresponding values of q, $= \frac{2\lambda - h}{2h - 2\lambda}$; the third the positions of the point of contact, say with the side $Z = 0$, or value of $X : Y$ calculated from the foregoing formula,

$$X \div Y = -1 + \frac{1}{2q^2} \pm \sqrt{\frac{1}{4q^4} - \frac{1}{q^2}};$$

and the fourth the apsidal distances, or say for the radius vector $X = Y$ the value of $Z : X$ calculated from the foregoing formula,

$$Z \div X = -2 + \frac{1}{q} + \frac{2}{q^2} \pm \frac{2}{q}\sqrt{\left(\frac{1}{q} - 1\right)\left(2 + \frac{1}{q}\right)}.$$

The Table is

	$\lambda : h$	q	Contact, $Z = 0$, $X : Y =$	Apses $X = Y$; $X : Z =$
*	·75	1·00	imposs.	1·
	·70	·6666	imposs.	·032 or 7·968
*	·666	·5	1·	0· : 16·
	·65	·4285	·320 or 3·124	·006 : 22·44
	·6	·25	·0721 : 13·9279	·059 : 67·941
	·55	·1111	·011 : 78·988	·15 : 337·85
*	·5	0·	0 or ∞	·25 : ∞
	·45	−·0909	·005 : 118·99	·39 : 457·61
	·4	−·1666	·029 : 33·971	·496 : 127·504
	·35	−·2308	·060 : 16·72	·648 : 61·79
*	·3	−·2857	·099 : 10·151	·812 : 37·187
*	·25	−·3333	·141 : 6·854	1· : 25
	·2	−·375	·207 : 4·904	1·218 : 17·89
	·15	−·4118	·276 : 3·622	1·48 : 13·25
	·10	−·4444	·372 : 2·690	1·813 : 9·937
	·05	−·4737	·515 or 1·941	3·15 or 6·46
*	0	−·5	1·	4·

where the asterisks show critical values of $\lambda : h$.

It is worth while to transform the equation to new coordinates X′, Y′, Z′ such that X′ = 0, Y′ = 0, Z′ = 0 represent the sides of the triangle formed by the three nodes. Writing for shortness X + Y + Z = P, YZ + ZX + XY = Q, XYZ = R, the equation is

$$(q^2\mathrm{P}-\mathrm{Q})^2 = 4\,(2q+1)\,\mathrm{PR}.$$

The expressions of X, Y, Z in terms of the new coordinates are of the form X′ + θP′, Y′ + θP′, Z′ + θP′, where P′ = X′ + Y′ + Z′, and writing also Q′ = Y′Z′ + Z′X′ + X′Y′, R′ = X′Y′Z′, then the values of P, Q, R are

$$(1+3\theta)\,\mathrm{P}',\quad \mathrm{Q}'+(2\theta+3\theta^2)\,\mathrm{P}'^2,\quad \mathrm{R}'+\theta\mathrm{P}'\mathrm{Q}'+(\theta^2+\theta^3)\,\mathrm{P}',$$

and the transformed equation is

$$[\{q^2(1+3\theta)^2-2\theta-3\theta^2\}\,\mathrm{P}'^2-\mathrm{Q}']^2 = 4\,(2q+1)(1+3\theta)\,\mathrm{P}\,\{(\theta^2+\theta^3)\,\mathrm{P}'^3+\theta\mathrm{P}'\mathrm{Q}'+\mathrm{R}'\},$$

which is satisfied by Q′ = 0, R′ = 0, if only

$$\{q^2\,(1+3\theta)^2-2\theta-\theta^2\}^2 = 4\,(2q+1)\,(1+3\theta)\,(\theta^2+\theta^3),$$

or if for a moment $q\,(1+3\theta) = \Omega$, the equation is

$$(\Omega^2-2\theta-\theta^2)^2 = 4\,(\theta^2+\theta^3)\,(2\Omega+1+3\theta),$$

that is, $\Omega^4-2\theta\,(\theta+2)\,\Omega^2-8\theta^2\,(1+\theta)\,\Omega,$

$$\Omega^4+\Omega^2\,(-6\theta^2-4\theta)+\Omega\,(-8\theta^3-8\theta^2)-3\theta^4-4\theta^3 = 0,$$

that is, $$(\Omega+\theta)^2\,(\Omega^2-2\theta\Omega-3\theta^2-4\theta) = 0,$$

and if the new axes pass through the nodes, then Ω + θ = 0; that is, $q\,(1+3\theta)+\theta = 0$, which equation gives the value of θ for which the new axes have the position in question; substituting in the first instance for q the value $\frac{-\theta}{3\theta+1}$, the equation becomes

$$\{2\theta\,(1+\theta)\,\mathrm{P}'^2+\mathrm{Q}'\}^2 = 4\,(1+\theta)\,\mathrm{P}'\{\theta^2\,(1+\theta)\,\mathrm{P}'^3+\theta\mathrm{P}'\mathrm{Q}'+\mathrm{R}'\},$$

that is, $$\mathrm{Q}'^2 = 4\,(1+\theta)\,\mathrm{P}'\mathrm{R}';$$

or, finally, substituting for θ its value in terms of q, the required equation is,

$$\mathrm{Q}'^2 = 4\,\frac{2q+1}{3q+1}\,\mathrm{P}'\mathrm{R}',$$

that is, $$(\mathrm{Y}'\mathrm{Z}'+\mathrm{Z}'\mathrm{X}'+\mathrm{X}'\mathrm{Y}')^2 = 4\,\frac{2q+1}{3q+1}\,\mathrm{X}'\mathrm{Y}'\mathrm{Z}'(\mathrm{X}'+\mathrm{Y}'+\mathrm{Z}').$$

In particular, for $q=0$ the equation is

$$(\mathrm{Y}'\mathrm{Z}'+\mathrm{Z}'\mathrm{X}'+\mathrm{X}'\mathrm{Y}')^2-4\mathrm{X}'\mathrm{Y}'\mathrm{Z}'\,(\mathrm{X}'+\mathrm{Y}'+\mathrm{Z}') = 0,$$

which is right, since, in the case in question (the tricuspidal curve), we have

$$\mathrm{X},\ \mathrm{Y},\ \mathrm{Z} = \mathrm{X}',\ \mathrm{Y}',\ \mathrm{Z}'.$$

I remark, in passing, that, taking the equation to be

$$(\mathrm{Y}'\mathrm{Z}'+\mathrm{Z}'\mathrm{X}'+\mathrm{X}'\mathrm{Y}')^2 = m\,\mathrm{X}'\mathrm{Y}'\mathrm{Z}'\,(\mathrm{X}'+\mathrm{Y}'+\mathrm{Z}'),$$

we may write herein $$Z' = \tfrac{1}{3} - x,$$
$$X' = \tfrac{1}{3} + \tfrac{1}{2}x - \frac{\sqrt{3}}{2}y,$$
$$Y' = \tfrac{1}{3} + \tfrac{1}{2}x + \frac{\sqrt{3}}{2}y;$$

where $$x = \frac{2\sqrt{m(m-3)}}{9}\cos\theta - \frac{2(m-3)}{9}\cos 2\theta,$$
$$y = \frac{2\sqrt{m(m-3)}}{9}\sin\theta + \frac{2(m-3)}{9}\sin 2\theta,$$

which are the formulæ for the description of the trinodal quartic as a unicursal curve.

I consider now the second position; viz., the horizontal plane now passes through the centre of the tetrahedron, and is parallel to two opposite edges. The equations of the nodal lines are here $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$; and if for convenience we assume the distance of the mid-points of opposite edges to be $=2$, or the half of this $=1$; then the equations of the faces will be

$$X = x+y+z-1 = 0,$$
$$Y = -x-y+z-1 = 0,$$
$$Z = x-y-z-1 = 0,$$
$$W = -x+y-z+1 = 0,$$

and the equation of the surface is

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{W} = 0;$$

viz., proceeding to rationalize, this is

$$X+Y+2\sqrt{XY} = Z+W+2\sqrt{ZW},$$

viz., $$2z+\sqrt{XY} = \sqrt{ZW};$$

we thence have $$4z^2+4z\sqrt{XY}+XY = ZW;$$

or since $$ZW-XY = 4z+4xy,$$

this is $$z+xy-z^2 = z\sqrt{XY};$$

whence $$(z+xy-z^2)^2 = z^2\{(z-1)^2-(x+y)^2\};$$

or reducing, $$2xyz+y^2z^2+z^2x^2+x^2y^2 = 0,$$

a form which puts in evidence the nodal lines. Considering z as constant, we have the equation of the section; this is a quartic having the node $(x=0,\ y=0)$, and two other nodes at infinity on the two axes respectively; moreover, the curve has for bitangents the intersections of its plane with the faces of the tetrahedron; or what is the same thing, attributing to z its constant value, the equations of the bitangents are

$$x+y+z-1 = 0,$$
$$-x-y+z-1 = 0,$$
$$x-y-z-1 = 0,$$
$$-x+y-z+1 = 0;$$

viz., these lines form a rectangle which is the section of the tetrahedron; observe that this is inscribed in the square the corners of which are $x=\pm 1$, $y=\pm 1$; viz., $z=+1$ (highest section), this is the dexter diagonal (considered as an indefinitely thin rectangle), and as z diminishes, the longer side decreases and the shorter increases until for $z=0$ (central section) the rectangle becomes a square; after which, for z negative it again becomes a rectangle in the conjugate direction, and finally, for $z=-1$ (lowest section) it becomes the sinister diagonal (considered as an indefinitely thin rectangle). But on account of the symmetry it is sufficient to consider the upper sections for which z is positive. The sides $\pm(x+y)+z-1=0$ parallel to the dexter diagonal of the square may for convience be termed the dexter sides, and the others the sinister sides. In what follows I write c to denote the constant value of z.

We require to know whether the bitangents have real or imaginary contacts; and for this purpose to find the coordinates of the points of contact.

Take first a dexter bitangent $x+y+c-1=0$; the coordinates of any point hereof are

$$x=\frac{1}{z}(1-c+\theta),\quad y=\frac{1}{z}(1-c-\theta),$$

where θ is arbitrary; and substituting in the equation of the curve, we should have for θ a two-fold quadric equation, giving the values of θ for the two points of contact respectively. We have

$$x^2+y^2=\tfrac{1}{2}\{(1-c)^2+\theta^2\},\qquad xy=\tfrac{1}{4}\{(1-c)^2-\theta^2\},$$

and thence

$$8c^2\{(1-c)^2+\theta^2\}+8c\{(1-c)^2-\theta^2\}+\{(1-c)^2-\theta^2\}^2=0,$$

viz., this equation is $\{\theta^2-(1-c)(1+3c)\}^2=0$,

a twofold quadric equation, as it should be; and the values of θ being $=\pm\sqrt{(1-c)(1+3c)}$, we see that these, and therefore the contacts, are real from $c=1$ to $c=-\frac{1}{3}$.

In exactly the same way for a sinister bitangent $\pm(x-y)-c-1=0$ we have

$x=\frac{1}{2}(1+c+\phi)$, $-y=\frac{1}{2}(1+c-\phi)$, and $\phi=\pm\sqrt{(1-3c)(1+c)}$, viz., the values of ϕ, and therefore the contacts, are real from $c=\frac{1}{3}$ to $c=-1$.

					Contacts of Dexter Bitangents.	Contacts of Sinister Bitangents.
That is,	$c=$	1	to	$\frac{1}{3}$	real	imaginary
	$c=$	$\frac{1}{3}$	to	$-\frac{1}{3}$	real	real
	$c=$	$-\frac{1}{3}$	to	-1	imaginary	real

or say $c=1$ to $\frac{1}{3}$, the contacts are real, imaginary; but $c=\frac{1}{3}$ to 0,

they are real, real. In the transition case, $c = \frac{1}{3}$, the sinister bitangents become osculating (4-pointic) tangents touching at points on the dexter diagonal. This can be at once verified.

Observe that when $c = 1$, we have

$$(x+y)^2 + x^2y^2 = 0,$$

so that the only real point is $x = 0$, $y = 0$; viz., this is a tacnode, having the real tangent $x + y = 0$. For $c = 0$ (central section) the equation becomes $x^2y^2 = 0$; viz., the curve is here the two nodal lines each twice.

It is now easy to trace the changes of form.

$c = 1$; curve is a tacnode, as just mentioned, tangent the dexter diagonal.

$c < 1 > \frac{1}{3}$; curve is a figure of 8 inside the rectangle, having real contacts with the dexter sides, but imaginary contacts with the sinister sides.

$c = \frac{1}{3}$; curve is a figure of 8 having real contacts with the dexter sides, and osculating (4-pointic) contacts with the sinister sides.

$c < \frac{1}{3} > 0$; curve is an indented figure of 8 having real contacts as well with the sinister as the dexter sides.

$c = 0$; curve is squeezed up into a finite cross, being the crunodal parts of the nodal lines; and joined on to these we have the acnodal parts, so that the whole curve consists of the lines $x = 0$, $y = 0$ each as a twofold line.

For tracing the curve, it is convenient to turn the axes through an angle of 45°; viz., writing $\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}$ in place of x, y respectively, the equation becomes

$$c(y^2-x^2) + c^2(y^2+x^2) + \tfrac{1}{4}(y^2-x^2)^2 = 0;$$

$x = 0$ gives $y^2 = 0$ or $y^2 = 4c(1-c)$,*

$y = 0$ gives $x^2 = 0$ or $x^2 = -4c(1+c)$.

Moreover, we have

$$4(c-c^2)(y^2-x^2) + 8c^2y^2 + (y^2-x^2)^2 = 0,$$

viz., $$(x^2-y^2+2c^2-2c)^2 = 4c^2\{(c-1)^2-2y^2\},$$

and similarly $$(y^2-x^2+2c^2+2c)^2 = 4c^2\{(c+1)^2-2x^2\},$$

putting in evidence the bitangents, now represented by the equations $c-1 = \pm y\sqrt{2}$ and $c+1 = \pm x\sqrt{2}$ respectively. And for the first of these, or $c-1 = \pm y\sqrt{2}$, we have for the points of contact $x^2 = \frac{1}{2}(1-c)(1+3c)$; and for the second of them, or $c+1 = \pm x\sqrt{2}$, the points of contact are $y^2 = \frac{1}{2}(1+c)(1-3c)$.

I consider the circumscribed cone having its vertex at a point $(0, 0, \gamma)$ on the nodal line $(x = 0, y = 0)$. Writing in the equation of

* Always imaginary when c is positive.

the surface $x = \lambda(z-\gamma)$, $y = \mu(z-\gamma)$, the equation, throwing out the factor $(z-\gamma)^2$, becomes

$$2\lambda\mu z + (\lambda^2+\mu^2)z^2 + \lambda^2\mu^2(z-\gamma)^2 = 0,$$

that is,

$$\begin{aligned}&(\lambda^2\mu^2+\lambda^2+\mu^2)z^2\\ &+2(-\gamma\lambda\mu+1)z\lambda\mu\\ &+\qquad \gamma^2 \;.\; \lambda^2\mu^2;\end{aligned}$$

and equating to zero the discriminant in regard to z, we have

$$\gamma^2(\lambda^2\mu^2+\lambda^2+\mu^2) - (-\gamma\lambda\mu+1)^2 = 0,$$

that is,

$$\gamma^2(\lambda^2+\mu^2) + 2\gamma\lambda\mu - 1 = 0;$$

and substituting herein the values $\lambda = \frac{x}{z-\gamma}$ and $\mu = \frac{y}{z-\gamma}$, we have the equation of the cone, viz., this is

$$\gamma^2(x^2+y^2) + 2\gamma xy - (z-\gamma)^2 = 0,$$

or what is the same thing,

$$\gamma^2(x^2+y^2-1) + 2\gamma(xy+z) - z^2 = 0;$$

viz., this is a quadric cone having for its principal planes $z-\gamma = 0$, $x+y = 0$, $x-y = 0$, these last being the planes through the nodal line and the two edges of the tetrahedron. In the particular case $\gamma = \infty$, the cone becomes the circular cylinder $x^2+y^2-1 = 0$.

The cone intersects the plane $z = 0$ in the conic

$$\gamma^2(x^2+y^2-1) + 2\gamma xy = 0,$$

which is a conic passing through the corners of the square $(x=0,\ y=\pm 1)$, $(x=\pm 1,\ y=0)$. For $\gamma > 1$, that is, for an exterior point, the conic is an ellipse having for the squares of the reciprocals of the semi-axes $1+\frac{1}{\gamma}$, $1-\frac{1}{\gamma}$ (this at once appears by writing in the equation $\frac{x+y}{\sqrt{2}}$, $\frac{x-y}{\sqrt{2}}$ in place of x, y respectively). In particular, for $\gamma = \infty$, the curve becomes the circle $x^2+y^2-1 = 0$. We have thus the apparent contour of the surface as seen from the point $z = \gamma$ on the nodal line, projected on the plane $z = 0$ of the other two nodal lines.

To find the curve of contact of the cone and surface, or say the surface-contour from the same point, write for a moment

$$\begin{aligned}\mathrm{V} &= \gamma(x^2+y^2-1) + 2\gamma(xy+z) - z^2,\\ \mathrm{U} &= (xy+z)^2 + z^2(x^2+y^2-1);\end{aligned}$$

then, substituting for x^2+y^2-1 its value in terms of V from the first equation, we find $\mathrm{U} = \left(xy+z-\frac{z^2}{\gamma}\right)^2 + \frac{z^2}{\gamma^2}\mathrm{V}$,

and the equations $\mathrm{U}=0$, $\mathrm{V}=0$ give therefore $xy+z-\frac{z^2}{\gamma} = 0$, or

say $\gamma(xy+z)-z^2=0$. The cone and surface therefore touch along the quadriquadric curve

$$\gamma^2(x^2+y^2-1)+2\gamma(xy+z)-z^2=0,$$
$$\gamma(xy+z)-z^2=0,$$

equations which may be replaced by

$$\gamma(x^2+y^2-1)+xy+z=0,$$
$$\gamma^2(x^2+y^2-1)+z^2=0.$$

In the case $\gamma=\infty$, the equations are $x^2+y^2-1=0$, $xy+z=0$, viz., the curve is the intersection of the hyperbolic paraboloid $xy+z=0$ by the cylinder $x^2+y^2-1=0$.

January 8th, 1874.

Dr. HIRST, F.R.S., President, in the Chair.

The visitors present were Messrs. Firth and S. H. Hodgson. Messrs. C. J. Lambert, M.A., and Mr. R. Forsyth Scott were elected Members; the Rev. James Booth, LL.D., F.R.S., Vicar of Stone, Bucks, and the Rev. W. H. Laverty, M.A., Fellow of Queen's College, Oxford, formerly junior Mathematical Scholar, Rector of Headley, Liphook, Hants, were proposed for election.

The thanks of the Society were voted to Mr. S. M. Drach for his present mentioned below.

The following communications were made:—

"On the Transformation of Continued Products into Continued Fractions:" Mr. J. W. L. Glaisher.

"The Foundations of the Differential Calculus and of Dynamics:" Prof. Clifford.

"A Method of Treating the Kinematical Question of the most general displacement of a Solid in Space:" Prof. Crofton, F.R.S.

"Link Trammels:" Mr. Perigal.

The following was taken as read:—

"On Hamilton's Characteristic Function for a Narrow Beam of Light:" Prof. J. Clerk-Maxwell, F.R.S.

The following presents were received:—

"Monatsbericht," Sept. and Oct., 1873.

"Pfaff's Disquisitiones Analyticæ," vol. i., Helmstadii, 1797 (no more printed): from Mr. Drach, F.R.A.S.

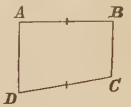
Prof. Crofton's communication was as follows:—

The following method of treating the kinematical theorem, that the most general displacement of a solid in space is equivalent to a

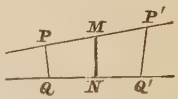
helical motion, is new to me (though possibly not really new), and has the great advantages over any other method I have seen, of the construction putting the motion palpably before the mind; and also of not merely proving the theorem, but finding, at the same time, the actual axis of rotation, giving the angle of rotation, and the amount of the translation along the axis—all in a most simple way.

I state one or two Lemmas in Solid Geometry which I make use of, for the proofs of which one does not know where to refer, but with which I will not trouble the Society. Some recognized system or treatise on Solid Geometry seems a great desideratum, I may remark *en passant.*

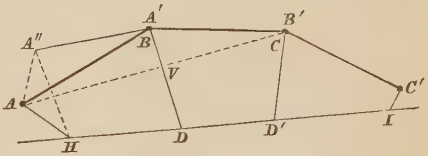
1. In a gauche quadrilateral, if B and C are right angles, and $\angle A = \angle D$, then $AB = DC$.



2. If MN is the shortest distance between two straight lines, and points P, P′ be taken on one of them, so that $MP = MP'$, and perpendiculars PQ, P′Q′, be let fall from P, P′ on the other, then $PQ = P'Q'$, $NQ = NQ'$, and also PQ, P′Q′ are inclined at equal angles to MN.



Let a solid be displaced in any way from a first position to a second; let A be any point of the solid in its first position; A′ the same point in the second position. Let B be that point of the solid (or rigidly connected with it) in its first position, which then was at A′, and now, in its second position, has moved to B′. Likewise, let C be that point which first was at B′, and now has moved to C′. Thus we have three points, A, B, C, of the solid, which have moved to A′, B′, C′; and the whole motion of the solid is of course fixed by that of these three points. Evidently



$$AB = BC = CC'; \text{ also } \angle ABC = \angle A'B'C',$$

(A, B, C, C′ are not in one plane).

Bisect the angles ABC, A′B′C′ *by the lines* BD, B′D′; *find the shortest distance* DD′ *between these bisectors. The solid can be brought from the first position to the second by a translation along the axis* DD′, *and a rotation round that axis. The amount of the translation* = DD′, *and the angle of rotation is the angle at which* B′D′ *is inclined to* BD.

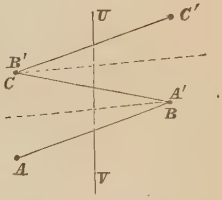
Produce DD′ both ways, drop perpendiculars AH, C′I on it from A and C′; in the gauche quadrilateral DBCD′ the angles at B, C are equal, therefore (Lemma I.) BD=B′D′. Join AC; it *cuts* BD at right angles in V; as VD is then the shortest distance between AC and HI, and VA=VC, therefore (Lemma II.) AH=CD′, HD=DD′, and AH is inclined to BD at the same angle as BD is to B′D′.

Likewise, by drawing A′C′, we can see finally that we have four

equal perpendiculars to the axis, AH, BD, B′D′, C′I; disposed at equal intervals HD = DD′ = D′I; and making successive equal angles with one another; hence a rotation round HI, through an angle equal to this common inclination, would bring HA to the position HA″, which is equal and parallel to A′D; and then a translation along the axis through the space HD or DD′ brings A″ to A′. In the same way, B, C are brought to B′, C′.

The same construction shows that, in the case where the displacement consists of a simple rotation round a point, as the translation DD′ must equal 0, the two bisectors BD, B′D′ must meet, and therefore A, B, C, C′ are in one plane. It thus gives, in this case, the simplest possible way of determining the axis of rotation; viz., taking the points A, B, C, C′ as above, let the two bisectors meet in a point O; the displacement is equivalent to a rotation round an axis through O at right angles to the plane ABCC′, and through an angle equal to the inclination of the two bisectors.

I should remark that there is an exceptional case to the above construction, viz., when the two bisectors are parallel. Here again all the figure is in one plane, and it is easy to see that the motion is equivalent to a translation $= \frac{1}{2}$AC, along an axis UV bisecting AB and BC, and *a rotation of* 180° round that axis.



The question of the displacement of a plane figure in its plane, is also much simplified by the above method. Let any point A have moved to B, and the point which *was* at B, to C; the centre of rotation is the point equi-distant from A, B, C.

February 12th, 1874.

Dr. HIRST, F.R.S., President, in the Chair.

The visitors present included Messrs. Herbert Spencer, G. H. Lewes, W. H. Corfield, and F. Pollock.

Mr. R. W. Genese, Mr. Robert Rawson, and the Rev. M. M. U. Wilkinson were admitted into the Society; the Revs. Dr. Booth and W. H. Laverty were elected Members; and Colonel A. R. Clarke, R.E., C.B., F.R.S., Mr. W. R. Browne, and Mr. Ernest Carpmael, B.A., Fellow of St. John's College, Cambridge, were proposed for election.

Prof. Clifford detailed at some length the views advanced in his paper "On the Foundations of Dynamics." A discussion took place in

which Messrs. Wilkinson, Moulton, Cayley, Roberts, and Lewes took part.

Mr. Clifford, having replied, proceeded to give an account of his paper on "The Free Motion of a Solid in Elliptic Space."

A paper by Mr. C. J. Monro, entitled "Note on the Inversion of Bernoulli's Theorem in Probabilities," was taken as read.

The following presents were received:—

"Bulletin," tome v., 2^{e} sémestre, 1873; and "Table des Matières," tome vi., Janvier, 1874.

"Monatsbericht," Nov. 1873.

"Transactions of the Connecticut Academy of Arts and Sciences," vol. ii., part 2: Newhaven, 1873.

"Vorlesungen über analytische Geometrie des Raumes:" Leipzig, 1869.—"Vorlesungen aus der analytischen Geometrie der geraden Linie, des Punctes und des Kreises in der Ebene:" Leipzig, 1873.—(Cloth, gilt edges): both presented by the Author, Dr. O. Hesse.

"Account of the Operations of the Great Trigonometrical Survey of India: vol. i., The Standards of Measure and the Base Lines. Also an Introductory Account of the Early Operations of the Survey during the period 1800—1830." By Colonel J. T. Walker, R.E., F.R.S.: Dehra Doon, 1870.—Presented by Her Majesty's Secretary of State for India.

March 12th, 1874.

Dr. HIRST, F.R.S., President, in the Chair.

Colonel A. R. Clarke, F.R.S., Mr. W. R. Browne, and Mr. Carpmael were elected Members of the Society; Prof. Charles Niven, M.A., Queen's College, Cork, and Mr. Thomas Muir, M.A., Assistant Professor of Natural Philosophy, Glasgow University, were proposed for election.

The following papers were read:—

"On certain Constructions for Bicircular Quartics," and "On a Geometrical Interpretation of the Equations obtained by equating to zero the Resultant and the Discriminant of two binary Quantics:" Prof. Cayley, F.R.S.

"On the Cartesian Equation of the Circle which cuts three given Circles at given Angles:" Mr. J. Griffiths.

"On another System of Poristic Equations:" Prof. Wolstenholme.

The following presents were received:—

"Bulletin," Fev. 1874.

The Third and Fourth Annual Reports of the Association for the Improvement of Geometrical Teaching.

"Ueber Schaaren von quadratichen Formen," von L. Kronecker; Berlin, 1874: from the Author.

"Monatsbericht," Dec. 1873.

Annual Report of the Board of Regents of the Smithsonian Institution.

"Bulletin de la Société Philomathique de Paris," tome dixième, Janvier—Juin, 1873: Paris, 1873.

"A Treatise on the Analytic Geometry of Three Dimensions," by George Salmon, D.D., F.R.S.; third edition, 1874: from the Author.

On certain Constructions for Bicircular Quartics.

By Prof. Cayley, F.R.S.

[*Read March 12th*, 1874.]

I call to mind that if F, G are any two points and F′, G′ their anti-points; then the circle on the diameter FG and that on the diameter F′G′ are concentric orthotomics, viz., they have the same centre, and the sum of the squared radii is = 0. Moreover, if the circles B, B′ are concentric orthotomics, and the circle A is orthotomic to B, then it is a bisector of B′, viz., it cuts B′ at the extremities of a diameter of B′; and B′ is then said to be a bifid circle in regard to A.

Given two real circles, these have an axial orthotomic, the circle, centre on the line of centres at its intersection with the radical axis, which cuts at right angles the given circles; viz., this axial orthotomic is real if the circles have no real intersection; but if the intersections are real, then the axial orthotomic is a pure imaginary, and instead thereof we may consider its concentric orthotomic, viz., this is the axial bifid of the two circles, or circle having its centre on the line of centres at the intersection thereof with the radical axis or common chord of the two circles, and having this common chord for its diameter.

If one of the circles is a pure imaginary, then we have still an axial orthotomic; viz., the pure imaginary circle is replaced by the concentric orthotomic; and the axial orthotomic is a bisector of the substituted circle; and so if each of the circles is a pure imaginary, then we have still an axial orthotomic, viz., each circle is replaced by the concentric orthotomic, and the axial orthotomic is a bisector of the substituted circles. And in either case the axial orthotomic of the original circles (one or each of them pure imaginary) *is real;* viz., this is given either

as the axial bisector of one real circle and orthotomic of another real circle; or as the axial bisector of two circles, from which the reality thereof easily appears; or we may verify it thus: Suppose that the two circles are $(x-\alpha)^2+y^2=\beta^2$, $(x-\alpha')^2+y^2=\beta'^2$, and their axial orthotomic $(x-m)^2+y^2=k^2$, then we have $(m-\alpha)^2=\beta^2+k^2$, $(m-\alpha')^2=\beta'^2+k^2$; subtracting, it appears that m is real; and then if either β^2 or β'^2 is negative, the equation containing this quantity shows that k^2 is positive; viz., the circle $(x-m)^2+y^2=k^2$ is real.

The above remarks have an obvious application to the theory of bicircular quartics; viz., a bicircular quartic is the envelope of a variable circle, having its centre on a conic, and orthotomic to a circle: it may be that this circle is a pure imaginary. We then replace it by the concentric orthotomic, and say that the curve is the envelope of a variable circle having its centre on a conic and bisecting a circle. We have thus a real form for cases which originally present themselves under an imaginary form.

The Bicircular Quartic with given vertices.

First, if the vertices are real; let the vertices taken in order be F, G, H, K.

First construction: On FG as diameter describe a circle, and on HK as diameter a circle; on the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ_1, and describe the axial orthotomic circle Σ_1 of the two circles (viz., the centre of Σ_1 is on the axis of symmetry at its intersection with the radical axis of the two circles); then the curve is the envelope of a variable circle having its centre on Θ_1 and orthotomic to Σ_1.

Second construction: On FH as diameter describe a circle, and on GK as diameter a circle. On the line terminated by the two centres (as transverse or conjugate axis(describe a conic Θ_2, and describe the axial bifid circle Σ_2' of the two circles (viz., the centre of Σ_2' is on the axis of symmetry at its intersection with the radical axis or common chord of the two circles, and its diameter is this common chord); then the curve is the envelope of a variable circle having its centre on Θ_2 and bisecting Σ_2'.

Third construction: On FK as diameter describe a circle, and on GH as diameter a circle; and then, as in the first construction, a conic Θ_3 and a circle Σ_3; the spiric is the envelope of a variable circle having its centre on Θ_3 and orthotomic to Σ_3.

Observe that in the three constructions the conics have always the same centre; and if the three conics are taken with the same foci, then the three constructions give one and the same bicircular quartic. The first and third constructions form a pair, and there is no reason for selecting one of them in preference to the other; but the second construction is unique; it is on this account natural to make use of it in discussing the series of curves with the given vertices.

In the particular case where the points F, G and H, K are situate symmetrically on opposite sides of a centre O (OF = OK, OG = OH), then in the third construction the centres each coincide with O, or the axis of the conic vanishes; hence the construction fails: the first and second constructions hold good, and in each of them the conic and circle are concentric. The curve is in this case quadrantal; having, besides the original axis of symmetry, another axis of symmetry through O, at right angles thereto.

Secondly, if the vertices are two real, two imaginary, say $f, g = \alpha \pm \beta\iota$; h, k. We modify the first or third construction; viz., if F′, G′ are the antipoints of F, G; then on F′G′ as diameter describe a circle, and on HK as diameter a circle. On the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ_1, and describe the axial bisector-orthotomic circle Σ_1 of the two circles; viz., this is the circle (centre on the axis of symmetry) which bisects the circle F′G′, and cuts at right angles the circle HK; then the curve is the envolope of the variable circle having its centre on Θ_1 and orthotomic to Σ_1.

Thirdly, if the vertices are all imaginary, say $f, g = \alpha \pm \beta\iota$; $h, k = \gamma \pm \delta\iota$, we modify the first or third construction. Take F′, G′ the antipoints of F, G, and H′, K′ the antipoints of H, K; then on F′G′ as diameter describe a circle, and on H′K′ as diameter a circle; on the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ, and describe the axial bisector-circle Σ of the two circles (viz., this is a circle, centre on the axis of symmetry, bisecting each of the circles): the curve is the envelope of a variable circle, centre on the conic Θ cutting at right angles the circle Σ.

A Geometrical Interpretation of the Equations obtained by equating to zero the Resultant and the Discriminants of two binary Quantics. By Prof. Cayley, F.R.S.

[*Read March 12th*, 1874.]

Consider the equations

$$U = (a, b, \ldots \mathcal{X} t, 1)^\lambda = 0,$$
$$U' = (a', b', \ldots \mathcal{X} t, 1)^{\lambda'} = 0;$$

and equating to zero the discriminants of the two functions respectively, and also the resultant of the two functions, let the equations thus obtained be

$$\Delta = (a, b, \ldots)^{2\lambda-2} = 0,$$
$$\Delta' = (a', b', \ldots)^{2\lambda'-2} = 0.$$
$$R = (a, b, \ldots)^{\lambda'} (a, b, \ldots)^\lambda = 0.$$

I take $(a, b, \ldots)$, $(a', b', \ldots)$ to be linear functions of the coordinates (x, y, z); and t to be an indeterminate parameter. Hence $U = 0$ represents a line the envelope whereof is the curve $\Delta = 0$, or what is the same thing, the equation $U = 0$ represents any tangent of the curve $\Delta = 0$; this is a unicursal curve of the order $2\lambda - 2$ and class λ, with $3(\lambda - 2)$ cusps and $\frac{1}{2}(\lambda - 2)(\lambda - 3)$ nodes. Similarly $U' = 0$ represents a line the envelope of which is the curve $\Delta' = 0$, which is a unicursal curve of the order $2\lambda' - 2$ and class λ', with $3(\lambda' - 2)$ cusps and $\frac{1}{2}(\lambda' - 2)(\lambda' - 3)$ nodes; the equation $U' = 0$ represents any tangent of this curve.

The equations $U = 0$, $U' = 0$, considered as existing simultaneously with the same value of t, establish a $(1, 1)$ correspondence between the tangents (or if we please, between the points) of the two curves. The locus of the intersection of the corresponding tangents is the curve $R = 0$, a unicursal curve of the order $\lambda + \lambda'$, with $\frac{1}{2}(\lambda + \lambda' - 1)(\lambda + \lambda' - 2)$ nodes and no cusps; consequently of the class $2(\lambda + \lambda' - 1)$.

It is to be shown that the curve $R = 0$ touches the curve $\Delta = 0$ in $\lambda' + 2\lambda - 2$ points, and similarly the curve $\Delta' = 0$ in $2\lambda' + \lambda - 2$ points.

It fact, consider any tangent T′ of the curve Δ'; let this meet the curve Δ in a point A, and let Q be the tangent at A to the curve Δ; suppose, moreover, that T is the tangent of Δ corresponding to the tangent T′ of Δ'. Then if Q and T coincide, the corresponding tangent of T′ will be Q, and the curve R will pass through A. It is easy to see that in this case the curves R, Δ will touch at A. Again, if P be a tangent from A to the curve Δ, then, if P and T coincide, the corresponding tangent of T′ will be P, and the curve R will pass through A; but in this case the point A will be a mere intersection, not a point of contact, of the two curves.

The tangents T, Q each correspond to T′, and they consequently correspond to each other. For a given position of T we have a single position of T′, and therefore $2\lambda - 2$ positions of A, or what is the same thing, of Q; that is, for a given position of T we have $2\lambda - 2$ positions of Q. Again, to a given position of Q corresponds a single position of A, therefore λ' positions of T′, therefore also λ' positions of T; that is, for a given position of Q we have λ' positions of T. The correspondence between T, Q is thus a $(\lambda', 2\lambda - 2)$ correspondence, and the number of united tangents is therefore $\lambda' + 2\lambda - 2$, or the curves R, Δ touch in $\lambda' + 2\lambda - 2$ points.

The tangents T, P each correspond to T′, and they therefore correspond to each other. For a given position of T we have a single position of T′, and therefore $2\lambda - 2$ positions of A, and thence $(2\lambda - 2)(\lambda - 2)$ positions of P; that is, for a given position of T we have $(2\lambda - 2)(\lambda - 2)$ positions of P. Again, to a given position of P corresponds $2\lambda - 4$ positions of A, therefore $(2\lambda - 4)\lambda'$ positions of

T′ or of T; that is, for a given position of P we have $(2\lambda-4)\lambda'$ positions of T. The correspondence between T, P is thus a $[2\lambda'(\lambda-2),\ 2(\lambda-1)(\lambda-2)]$ correspondence, and the number of united tangents is $2(\lambda+\lambda'-1)(\lambda-2)$; or the curves R, Δ meet in $2(\lambda+\lambda'-1)(\lambda-2)$ points.

Reckoning the contacts twice, the total number of intersections of R, Δ is $2\lambda'+4\lambda-4+2(\lambda+\lambda'-1)(\lambda-2), = (\lambda+\lambda')(2\lambda-2)$, as it should be.

In the particular case $\lambda=\lambda'=2$, the curves Δ, Δ′ are conics, and the curve R is a quartic curve touching each of the conics 4 times; this is at once verified, since the equations here are $ac-b^2=0$, $a'c'-b'^2=0$, $4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0$.

On the Cartesian Equation of the Circle which cuts three given Circles at given Angles. By John Griffiths, M.A.

[*Read March 12th*, 1874.]

The equation of a circle cutting three given ones at known angles may be at once obtained from the following proposition:

Let P be any point on a circle U cutting a given circle S_1, whose centre is C, at the angle θ_1; Q the point on U diametrically opposite to P; p the length of the perpendicular from Q upon the polar of P with respect to S_1; then

$$p\times(\text{distance CP}) = -2Rr_1\cos\theta_1,$$

R and r_1 being the radii of U and S_1 respectively.

This property admits of an easy proof by ordinary geometry, and may also be deduced in the following manner from the Cartesian equations of the two circles.

If we write

$S=x^2+y^2-r^2=0$, $U=(x-\alpha)^2+(y-\beta)^2-R^2=0$, where $\alpha^2+\beta^2=\delta^2$;

perpendicular from (x, y) on polar of (ξ, η) in regard to S is

$$=\frac{\xi x+\eta y-r^2}{\sqrt{\xi^2+\eta^2}};$$

i. e., perpendicular from $(\xi_1, \eta_1) = \dfrac{\xi\xi_1+\eta\eta_1-r^2}{\sqrt{\xi^2+\eta^2}}$;

and if (ξ_1, η_1) is opposite to (ξ, η), each being on circle U,

$$(\xi-\alpha)^2+(\eta-\beta)^2-R^2=0,\quad \xi_1=2\alpha-\xi,\quad \eta_1=2\beta-\eta,$$

$$p\sqrt{\xi^2+\eta^2}=\xi(2\alpha-\xi)+\eta(2\beta-\eta)-r^2=\delta^2-R^2-r^2=-2Rr\cos\theta.$$

Putting, then, the equations of the given circles in the homogeneous forms

$$S_1 = x^2+y^2+2g_1xz+2f_1yz+c_1z^2 = 0,$$
$$S_2 = x^2+y^2+2g_2xz+2f_2yz+c_2z^2 = 0,$$
$$S_3 = x^2+y^2+2g_3xz+2f_3yz+c_3z^2 = 0,$$

and supposing that (x, y), (x', y') are two diametrically opposite points on the circle which cuts S_1, S_2, S_3 at the angles θ_1, θ_2, θ_3, respectively, we get immediately from the above,

$$x'\frac{dS_1}{dx}+y'\frac{dS_1}{dy}+\frac{dS_1}{dz} = -4Rr_1\cos\theta_1,$$
$$x'\frac{dS_2}{dx}+y'\frac{dS_2}{dy}+\frac{dS_2}{dz} = -4Rr_2\cos\theta_2,$$
$$x'\frac{dS_3}{dx}+y'\frac{dS_3}{dy}+\frac{dS_3}{dz} = -4Rr_3\cos\theta_3;$$

hence, by eliminating x', y', the equation of the required circle U may be written

$$\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_1}{dy}, & \frac{dS_1}{dz}+4Rr_1\cos\theta_1 \\ \frac{dS_2}{dx}, & \frac{dS_2}{dy}, & \frac{dS_2}{dz}+4Rr_2\cos\theta_2 \\ \frac{dS_3}{dx}, & \frac{dS_3}{dy}, & \frac{dS_3}{dz}+4Rr_3\cos\theta_3 \end{vmatrix} = 0;$$

viz., this is

$$\Omega+4R\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_1}{dy}, & r_1\cos\theta_1 \\ \frac{dS_2}{dx}, & \frac{dS_2}{dy}, & r_2\cos\theta_2 \\ \frac{dS_3}{dx}, & \frac{dS_3}{dy}, & r_3\cos\theta_3 \end{vmatrix} = 0,$$

where $\Omega = 0$ is the equation of the orthotomic circle.

The algebraical solution of the problem will, therefore, be complete if we can determine the value of R, the radius of U.

This can be done by using Prof. Cayley's equation connecting the mutual distances of four points in a plane. The result is a quadratic in R; viz.,

$$(a+b+c+2f+2g+2h)\,R^2$$
$$-2\{(a+h+g)\,r_1\cos\theta_1+(h+b+f)\,r_2\cos\theta_2+(g+f+c)\,r_3\cos\theta_3\}\,R$$
$$+A'B'C'+2F'G'H'-A'F'^2-B'G'^2-C'H'^2 = 0;$$

where a, b, c, &c., are the invariant quantities given in a paper by the present writer, which was published in Vol. III., No. 37, of the Proceedings of the Mathematical Society.

It may be observed that some of the properties of the circle U, given by Plücker in his solution of the present problem, may be readily de-

duced from the foregoing equation. For instance, the radical axis of U and J, the Jacobian of the circles S_1, S_2, S_3, is the line

$$\left(\frac{dS_2}{dx}\frac{dS_3}{dy} - \frac{dS_2}{dy}\frac{dS_3}{dx}\right) r_1 \cos\theta_1 + \ldots\ldots = 0;$$

or, if we use areal coordinates x, y, z, and take the triangle formed by joining the centres of the given circles for a triangle of reference,

$$r_1 \cos\theta_1 x + r_2 \cos\theta_2 y + r_3 \cos\theta_3 z = 0;$$

i.e., the radical axis in question coincides with an axis of similitude of three circles of radii $r_1 \cos\theta_1$, $r_2 \cos\theta_2$, $r_3 \cos\theta_3$, and concentric with S_1, S_2, S_3, respectively. The above method will also give us the equation of a sphere which cuts four given spheres at given angles.

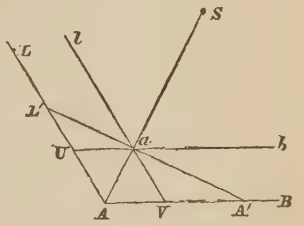
On another System of Poristic Equations.

By Prof. Wolstenholme, M.A.

[*Read March 12th*, 1874.]

If there be two similar and similarly situated polygons of n sides, ABC ... KL, *abc ... kl*, and if through a we draw a straight line L′A′ terminated by LA, AB, and then draw A′*b*B′, B′*c*C′, ... K′*l*L″, terminated by BC, CD, ... KL, then to every position of L′ there is one and only one position of L″, and to every position of L″ one and only one of L′, so that if $AL' = x'$ and $AL'' = x''$, x', x'' must be connected by an equation of the form $Ax'x'' + Bx' + Cx'' + D = 0$. Hence in general there will be two positions of L′ for which L″ will coincide with it, *i.e.*, in general there are two and only two polygons inscribed in the given polygon ABC ... KL, and circumscribed to the polygon *abc ... kl*. It may, however, under certain circumstances, happen that L″ always coincides with L′, and the conditions for this I proceed to investigate.

Let $\frac{AA'}{AB} = x_1$, $\frac{AL'}{AL} = 1 - x_n$; produce *ba*, *la* to meet AL, AB in U, V; then since the polygons are similar and similarly situated, $\frac{AU}{AL} = \frac{AV}{AB} = k$ suppose. We shall then have for L′A′ passing through a,



$$\frac{k}{x_1} + \frac{k}{1-x_n} = 1, \text{ or } \frac{1}{x_1} + \frac{1}{1-x_n} = \frac{1}{k};$$

and for an in- and circum-scribed polygon the system of equations

$$\frac{1}{x_1} + \frac{1}{1-x_n} = \frac{1}{x_2} + \frac{1}{1-x_1} = \frac{1}{x_3} + \frac{1}{1-x_3} = \ldots = \frac{1}{x_n} + \frac{1}{1-x_{n-1}} = \frac{1}{k}.$$

Hence $$1-x_1=\frac{k}{1-\dfrac{k}{x_2}},\quad \text{or}\quad x_1=1-\frac{k}{1-}\,\frac{k}{x_2};$$

so $$x_2=1-\frac{k}{1-}\,\frac{k}{x_3},$$

and so on until we have $$x_n=1-\frac{k}{1-}\,\frac{k}{x_1},$$

so that the equation for x_1 is the (generally) quadratic

$$x_1=1-\frac{k}{1-}\,\frac{k}{1-}\,\frac{k}{1-}\,\ldots\ldots-\frac{k}{1-}\,\frac{k}{x_1}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

there being $2n$ of the k's. Let $\frac{p_r}{q_r}$ be the rth convergent to the infinite continued fraction $1-\frac{k}{1-}\,\frac{k}{1-}\,\ldots\ldots$, of which the first three are $\frac{1}{1}$, $\frac{1-k}{1}$, $\frac{1-2k}{1-k}$, and since both numerators and denominators are formed according to the law $u_r=u_{r-1}-ku_{r-2}$, we shall always have $p_r=q_{r+1}$. Also we shall have

$$p_r=\mathrm{A}\alpha^{r-1}+\mathrm{B}\beta^{r-1},\ \text{where } \alpha,\beta \text{ are the roots of } z^2-z+k=0,$$

and $$1=\mathrm{A}+\mathrm{B},\quad 1-k=\mathrm{A}\alpha+\mathrm{B}\beta,$$

therefore $$\mathrm{A}=\frac{1-k-\beta}{\alpha-\beta}=\frac{\alpha-\alpha\beta}{\alpha-\beta}=\frac{\alpha^2}{\alpha-\beta},$$

and $\mathrm{B}=\dfrac{\beta^2}{\beta-\alpha}$, hence $p_r=\dfrac{\alpha^{r+1}-\beta^{r+1}}{\alpha-\beta}$; and therefore $q_r=\dfrac{\alpha^r-\beta^r}{\alpha-\beta}$.

But the final fraction of the continued fraction (1) is

$$\frac{x_1p_{2n}-kp_{2n-1}}{x_1q_{2n}-kq_{2n-1}}\quad\text{or}\quad\frac{x_1p_{2n}-kp_{2n-1}}{x_1p_{2n-1}-kp_{2n-2}},$$

or the equation for x_1 is

$$x_1^2p_{2n-1}-x_1(p_{2n}+kp_{2n-3})+kp_{2n-1}=0,$$

or since $$p_{2n}=p_{2n-1}-kp_{2n-2},\quad p_{2n-1}(x_1^2-x_1+k)=0,$$

and the system of equations is poristic if only $p_{2n-1}=0$, *i.e.*, if $\alpha^{2n}=\beta^{2n}$.

Putting $1-4k=-\tan^2\theta$, $\alpha=\dfrac{1+\sqrt{1-4k}}{2}=\dfrac{\cos\theta+\sqrt{-1}\,\sin\theta}{2\cos\theta}$,

and the final equation is $\dfrac{\sin 2n\theta}{2^{n-1}\cos^{2n}\theta}=0$, which gives $\theta=\dfrac{r\pi}{2n}$,

r being any integer from 1 to $n-1$, 0 being excluded because that gives $k=\frac{1}{4}$, and makes α, β equal, in which case the form of our assumption for p must be altered. This point will be considered afterwards. $r=n$ makes $\sin 2n\theta=0$, but makes also $\cos\theta=0$, and the equation for θ is not satisfied. Integers from $n+1$ to $2n-1$ satisfy the equation for θ, but give no new values for k. There are therefore, on the whole, $n-1$ values of k for which the system is poristic. In each

case there will be an infinite number of in- and circum-scribed polygons, but their areas are all equal to each other in any one case, being a mean proportional between the areas of the two polygons.

If k have any value other than these, we must have

$$x_1^2 - x_1 + k = 0, \quad \text{or} \quad \frac{1}{x_1} + \frac{1}{1-x_1} = \frac{1}{k},$$

whence $x_2 = x_1$, and similarly $x_3 = x_2$, ...,

or $$x_1 = x_2 = \ldots\ldots = x_n = \frac{1 \pm \sqrt{1-4k}}{2}.$$

These will only be real if $k < \frac{1}{4}$, whereas all the values of k for poristic systems are $> \frac{1}{4}$; or if k have one of the values for which the system is poristic, the two normal in- and circum-scribed polygons do not exist. Since $\alpha + \beta = 1$, we shall have, if A′B′C′... K′L′ and A″B″C″... K″L″ be the two polygons, AA′ = A″B, BB′ = B″C, and so on; so that when the given polygons are regular the two in- and circum-scribed polygons are equal in all respects.

Let us now consider the case of $k = \frac{1}{4}$, in which case $\alpha = \beta$, and we shall have the two normal in- and circum-scribed polygons coincident bisecting each side of the polygon ABC ... KL. The form to be taken for p_r is then $[A + B(r-1)]\frac{1}{2^{r-1}}$, and $1 = A$, $\frac{3}{4} = \frac{A+B}{2}$, therefore $B = \frac{1}{2}$, and $p_r = \frac{r+1}{2^r}$. The final equation will then be

$$\frac{2n}{2^{2n-1}}\left(x_1^2 - x_1 + \frac{1}{4}\right) = 0,$$

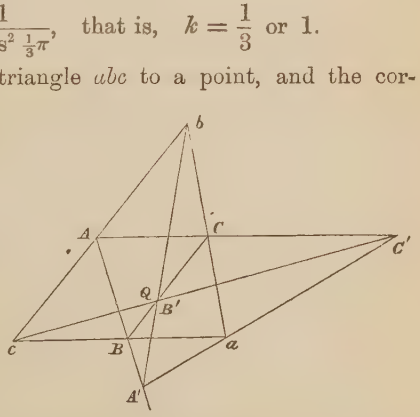
so that we cannot have $p_{2n-1} = 0$, and we have the single solution

$$x_1 = x_2 = \ldots\ldots = x_n = \tfrac{1}{2}.$$

If $n = 3$, we shall have a poristic system for

$$k = \frac{1}{4\cos^2 \frac{1}{6}\pi} \quad \text{or} \quad \frac{1}{4\cos^2 \frac{1}{3}\pi}, \quad \text{that is,} \quad k = \frac{1}{3} \text{ or } 1.$$

The former value reduces the triangle abc to a point, and the corresponding in- and circum-scribed triangle to a straight line. The latter value makes A, B, C the middle points of the sides of the triangle abc, and we have the following geometrical theorem :—A, B, C being the middle points of the sides of the triangle abc, if any straight line through a meet AB, AC in A′, C′, A′b and cC′ will intersect each other (in B′) in a point lying on BC, and the area of the triangle A′B′C′ will



be twice that of the triangle ABC. This is not likely to be a novelty. For this case it would be more symmetrical to write A′, B′, C′ for B′, C′, A′. The following theorems are also true of the figure (when this change of lettering is made):—A′a, B′b, C′c are parallel to each other, and AA′, BB′, CC′ meet in a point.

When $n=4$, we have the three values for k, $\frac{1}{4\cos^2\frac{1}{8}\pi}$, $\frac{1}{4\cos^2\frac{2}{8}\pi}$, $\frac{1}{4\cos^2\frac{3}{8}\pi}$, of which the middle one reduces the quadrilateral $abcd$ to a point, and the in- and circum-scribed polygon to a straight line; also ABCD must be a parallelogram. The other two values supply proper solutions; but the in- and circum-scribed polygon can never, as far as I have investigated, be a convex quadrilateral when the two given ones are squares or parallelograms. Taking them squares, we shall have the following theorem:—If ABCD, $abcd$ be two concentric squares with their sides parallel and the side of one equal to the sum of a side and diagonal of the other, we can find an infinite number of quadrangles A′B′C′D′ such that A′, B′, C′, D′ lie on the sides AB, BC, CD, DA, and their sides A′B′, B′C′, C′D′, D′A′ pass through b, c, d, a respectively; or such that A′, B′, C′, D′ lie on the sides ab, bc, cd, da, and their sides A′B′, B′C′, C′D′, D′A′ pass through D, A, B, C respectively. The area of any such quadrangle will be a mean proportional between those of the squares. I have not considered the particular cases for other values of n.

April 9th, 1874.

Prof. CAYLEY, F.R.S., Vice-President, in the Chair.

Mr. J. Hammond, B.A., Scholar of Queens' College, Cambridge, was proposed for election; the Rev. Prof. Townsend, F.R.S., was admitted into the Society; and Prof. C. Niven and Mr. T. Muir were elected Members of the Society.

Mr. G. H. Darwin, M.A., read a paper "On Probable Error in Statistics," with a Note on "The Combination of Statistics."

Mr. Glaisher made remarks on the paper, referring the Author to a treatment of the question by Laplace and De Morgan.

Mr. Merrifield, F.R.S., next gave an account of his communication entitled "Determination of the Form of the Dome of Uniform Stress."

Mr. A. J. Ellis, F.R.S., spoke on the subject of "Algebra considered as the Calculus of Similar Triangles."* Prof. H. J. Smith, F.R.S.,

* Mr. Ellis has given his views at length in a work entitled, "Algebra Identified with Geometry;" in a series of five tracts. i. Euclid's conception of Ratio and Proportion. ii. "Carnot's Principle" for Limits. iii. The Laws of Tensors. iv. The Laws of Clinants. v. Stigmatic Geometry; or the Correspondence of Points in a Plane. A copy of this work (84 pp. 8vo.) has been presented to every member of the London Mathematical Society.

gave a supplementary Note to his former paper "On the Higher Singularities of Plane Curves." A paper by Mr. H. M. Taylor, M.A., "On Inversion, with special reference to the Inversion of an Anchor-Ring," was taken as read.

The following presents were received:—

"Crelle," 77 Band, 2es and 3es Heft, Berlin, 1874.

"Monatsbericht," Januar., 1874.

"On the Geometrical Isomorphism of Crystals, and the derivation of all other forms from those of the Cubical System," by the Rev. W. Mitchell, M.A.: from the President and Council of the Victoria Institute.

May 14*th*, 1874.

Dr. HIRST, F.R.S., President, in the Chair.

Mr. J. Hammond, B.A., was elected a Member; the Rev. A. J. Stevens, M.A., Fellow of St. John's College, Cambridge, and Mr. W. Ritchie, were proposed for election.

The President, having vacated the Chair, gave an account of his paper "On the Correlation of Two Planes." Mr. Spottiswoode (the Chairman *pro tem.*) and Prof. Clifford spoke on the subject of the paper. Mr. Spottiswoode, F.R.S., then spoke briefly on the subject of his communication, "The Contact of Quadrics with other Surfaces." Remarks were made on the paper by the Chairman and Prof. Clifford. A paper by Mr. J. H. Röhrs, M.A., Jesus College, Cambridge (communicated by Prof. Cayley), on a "Rotating Sphere filled with Viscous Fluid," was taken as read.

The following presents were received:—

"Bulletin," Mars, Avril, 1874.

"Monatsbericht," Fev., 1874.

"Annali di Matematica," tom. vi., fasc. i., Dec., 1873.

Des Cartes. "Principia Matheseos universalis, seu Introductio ad Geometriæ methodum Renati Des Cartes conscripta ab Er. Bartholino Casp. Fil. editio tertia." Amstelod., 1683.

"Neutoni genesis curvarum per umbras seu perspectivæ universalis elementa exemplis coni sectionum et linearum tertii ordinis illustrata." Londini, 1746.

"An Essay on Magnetic Attraction, particularly as respects the Deviation of the Compass on Shipboard, occasioned by the Local Influence of the Guns, &c." By Peter Barlow. London, 1820.

"Miscellanea Curiosa: containing a Collection of some of the Principal Phenomena in Nature, accounted for by the Greatest Philosophers of this Age, being the most valuable Discourses read and delivered to the Royal Society." In 3 vols. 2nd edition. 1708.

These four Works presented to the Society by Mr. S. M. Drach, F.R.A.S., from the library of Mr. John Smith.

"Proceedings of the Literary and Philosophical Society of Manchester." Vol. viii. (1868-9), vol. ix. (1869-70), vol. x. (1870-71.)

"Memoirs of the Literary and Philosophical Society of Manchester." Third series, fourth volume, 1871.

"Jahrbuch über die Fortschritte der Mathematik." Dritter Band, Heft 3. Jahrgang, 1871.

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*On the Correlation of two Planes.** *By* Prof. Hirst, F.R.S.

[*Read May 14th*, 1874.]

Definition and Determination of a Correlation.

1. A correlation is said to be established between two planes, when their points and right lines are so associated, that to each point in one of the planes, and to each line passing through that point, respectively correspond, in the other plane, *one* line and *one* point in that line.

2. It will be convenient to apply the familiar terms *pole* and *polar* to a point and its corresponding right line.

3. It has been shown by Chasles† and others, that such a correlation may be established, and that in one way only, when four points, in one plane, and their polars, in the other plane, are given; provided always that no three of the points are collinear, and no three polars concurrent. In fact, if $A_1\,B_1\,C_1\,D_1$ be the four points in the first plane, and $a_2\,b_2\,c_2\,d_2$ their respective polars in the other, then, x_2 being the polar of any fifth point X_1, it follows at once from the definition of a correlation that the pencils $A_1\,(B_1\,C_1\,D_1\,X_1)$ and $B_1\,(A_1\,C_1\,D_1\,X_1)$ must be equi-anharmonic, respectively, with the rows $a_2\,(b_2\,c_2\,d_2\,x_2)$ and $b_2\,(a_2\,c_2\,d_2\,x_2)$. Hence, by a well-known method, if X_1 be given, the intersections of its polar x_2 with a_2 and b_2, and hence x_2 itself, are readily found.

4. From the definition of a correlation it also follows that if the polar of A_1 pass through A_2, the polar of A_2 will pass through A_1; and similarly, that if the pole of a_1 lie on a_2, the pole of a_2 will lie in a_1. Hence we may term A_1 and A_2 *conjugate points*, and a_1 and a_2 *conjugate lines* of the correlation.

5. That in any correlation two given points, or two given right lines,

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* Of the two questions, Correlation and Homography, the first has been selected for special consideration in this paper. With suitable modifications, however, as shown in the foot-notes on pages 61 and 62, the results are all applicable to the second question.

† "Géométrie Supérieure," art. 581.

shall be conjugate to each other, is obviously to be regarded as one condition; that a given point and line shall bear to each other the relation of pole and polar is, accordingly, equivalent to two conditions. Hence, and from art. 3, we may infer that eight conditions are necessary and sufficient for the establishment of a correlation between two planes.*

6. The problem to determine a correlation between two planes which shall satisfy *any* eight given conditions, is susceptible, in general, of a finite number of solutions. The determination of this number, when the conditions are of the elementary kind, single and double, described in the last article,—a necessary step towards the solution of the more general problem,—is one of the objects of the present paper.

Systems of Correlations.

7. Two planes may obviously be correlated in innumerable ways so as to satisfy *seven* given conditions; the totality of such correlations constitute what may be termed *a system of correlations satisfying seven conditions.*

For example, if the polars $p_2\ q_2\ r_2$ of three points $P_1\ Q_1\ R_1$ were given, as well as two conjugate points A_1 and A_2,—a set of seven conditions which may be conveniently indicated thus:

$$\left.\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

then, in general, any line a_2 passing through A_2 may be regarded as the polar of A_1, and a correlation established, in the manner indicated in art. 3, which shall satisfy the eight conditions,

$$\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & a_2 \end{matrix}$$

and therefore, *à fortiori*, the seven given conditions (1).

By giving to a_2 all possible positions, we obtain the several correlations of the system. We shall find it convenient to denote such a system by the symbol $\left(\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{matrix}\right)$.

8. It should be observed, however, that whenever a_2 passes through the intersection of any two of the three given lines $p_2\ q_2\ r_2$, the provision alluded to in art. 3 ceases to be observed, and no correlation, in the ordinary sense of the term, can be established.

9. Such exceptions occur in every system of correlations. As

* This also follows immediately from the "*expression analytique des figures corrélatives*," given by Chasles in the "Géométrie Supérieure," art. 594. Expressed in trilinear co-ordinates, the result there arrived at is that $\lambda_2 x + \mu_2 y + \nu_2 z = 0$ will be the polar of $\alpha_1\beta_1\gamma_1$ if

$$\lambda_2 : \mu_2 : \nu_2 = l\alpha_1 + m\beta_1 + n\gamma_1 : l'\alpha_1 + m'\beta_1 + n'\gamma_1 : l''\alpha_1 + m''\beta_1 + n''\gamma_1,$$

where the ratios of $l, m, n,\ l', m', n',\ l'', m'', n''$ are the eight constants of the correlation.

another example we may take the system

$$\begin{pmatrix} P_1 & Q_1 & R_1 & a_1 \\ p_2 & q_2 & r_2 & a_2 \end{pmatrix},$$

wherein a_1 and a_2 denote given conjugate lines. The several correlations of this system are obviously obtained by giving to A_1, the pole of a_2, all possible positions on a_1; but whenever A_1 comes into line with two of the three points $P_1\,Q_1\,R_1$, the method of establishing a correlation described in art. 3 ceases to be applicable.

10. Instead of excluding cases such as those described in the last two articles, it will be found of the highest importance to admit them as *exceptional correlations* into the system satisfying seven conditions, and carefully to study their properties. The part they play in the general theory of correlation will be found to be strictly analogous to that played by degenerate conics in the investigations of Chasles on systems of conics satisfying four conditions.*

Origin and Nature of Exceptional Correlations.

11. With a view of obtaining an insight into the nature and origin of exceptional correlations, we will first consider all the exceptional forms which a homographic relation between two planes may assume; or, what is equally general, all the exceptional modes of putting two planes Π_1, Π' into perspective with each other.

It is clear that, so long as the centre of perspective lies in neither of the two planes, the homographic relation between them is of the ordinary type; we have, therefore, merely to consider the cases where the centre of perspective lies, 1) in one of the two planes, and 2) in the intersection of the two planes.

12. If the centre of perspective be a point Σ_1 in the plane Π_1, and if σ' be the line in which the second plane Π' is intersected by Π_1, it is obvious, *first*, that to every point M' of Π', which is situated on σ', corresponds, in Π_1, an indeterminate point on the line $\overline{\Sigma_1 M'}$, whilst to every other point of Π' corresponds the point Σ_1 itself; and, *secondly*, that to the point Σ_1 corresponds a wholly indeterminate point of the plane Π', whilst to every other point M_1 of Π_1 corresponds the intersection M' of $\overline{\Sigma_1 M_1}$ and σ'.

With respect to the correspondence between the lines of the two planes, it is equally obvious, *first*, that to the line σ' in Π' corresponds a wholly indeterminate line in Π_1, whilst to every other line m' in Π' corresponds the line through Σ_1 and $(m'\sigma')$; and, *secondly*, that to every line m_1 in Π_1, which passes through Σ_1, corresponds an indeterminate line in Π' passing through the intersection $(m_1\sigma')$, whilst to every other line in Π_1 corresponds the line σ' itself.

13. If the centre of perspective Σ_1 lie in the intersection σ' of the two

* "Comptes rendus des séances de l'Académie des Sciences," 1864.

planes, Π_1 and Π', then to Σ_1, regarded as a point of either plane, corresponds a wholly indeterminate point of the other plane; to any other point of σ', in either plane, corresponds an indeterminte point of the same line σ' in the other plane; whilst to every other point, of either plane, corresponds the point Σ_1 itself.

The lines of the two planes correspond in the following manner: to σ', regarded as a line of either plane, corresponds a wholly indeterminate line in the other plane; to any other line through Σ_1, in either plane, corresponds an indeterminate line through Σ_1 in the other plane; whilst to any other line, in either plane, corresponds the line σ' itself.

14. Passing now from the perspective, to any other position of the two planes, we conclude that the *homographic correspondence between them may assume either of the following two exceptional or singular forms.*

First. There may be a singular point in one plane, and a singular line in the other, whose respective correspondents are wholly indeterminate. To each point in the singular line will then correspond an indeterminate point in a determinate line passing through the singular point, whilst to each such line through the singular point will correspond an indeterminate line passing through a determinate point of the singular line.

It is of importance to observe that between the above-mentioned points on the singular line and the associated lines through the singular point there must, by art. 12, always exist an equi-anharmonic, or (1, 1) correspondence.

Secondly. In each plane there may be a singular line, and a singular point situated in that line, whose respective correspondents are wholly indeterminate. To every other point in a singular line will then correspond an indeterminate point in the other singular line, whilst to every other line passing through a singular point will correspond an indeterminate line through the other singular point.

15. If an exceptional homographic relation exist between two planes Π_1 and Π', and any ordinary correlation, such as that described in art. 3, be established between the latter, and a third plane Π_2, it is evident that between the former Π_1 and this third plane Π_2 an exceptional correlation will exist; and *vice versâ*, from any exceptional correlation, we can always pass, by means of an auxiliary ordinary correlation, to an exceptional homographic relation. This principle enables us, readily, to deduce *all possible exceptional forms of correlation* from the results of the last article.

16. In so doing we must consider three cases, since the first case of the last article presents two varieties.

First. The exceptional homographic relation between Π_1 and Π' is such that in Π_1 there is a singular point Σ_1, and in Π' a singular line σ'. The pole of σ', in the plane Π_2, being Σ_2, we shall have, between Π_1 and Π_2, an *exceptional correlation with singular points* Σ_1 and Σ_2, whose characteristic properties, easily deducible from art. 14, may be briefly described thus:

The polar of a singular point is wholly indeterminate. The pole of every

line through a singular point is an indeterminate point on a determinate line through the other singular point, the two lines thus associated being always corresponding rays of equi-anharmonic pencils.

Secondly. The exceptional homographic relation between Π_1 and Π' is such, that there is a singular line σ_1 in Π_1, and a singular point Σ' in Π'.

The polar of Σ', in the plane Π_2, being σ_2, we shall now have, between the planes Π_1 and Π_2, *an exceptional correlation with singular lines* σ_1 and σ_2, the characteristic properties of which, as deduced from art. 14, will be as follows :

The pole of a singular line is wholly indeterminate. The polar of every point in a singular line is an indeterminate line through a determinate point of the other singular line, the two points thus associated being always corresponding points of two equi-anharmonic rows.

Thirdly. The exceptional homographic relation between Π_1 and Π_2 is such that in each of these planes there is a singular line, and a singular point situated on that line.

Between the planes Π_1 and Π_2 we shall now have an *exceptional correlation with singular lines and points* (in each line a point), of which the following properties are characteristic :

The pole of each singular line, as well as the polar of each singular point, is wholly indeterminate. The polar of any point in a singular line, not coincident with the singular point situated therein, is an indeterminate line through the other singular point.

An exceptional correlation with singular points is determined when the positions of those points, and of three pairs of conjugate lines passing through them are known. In like manner, an exceptional correlation with singular lines requires, for its determination, a knowledge of the positions of those lines, as well as of three pairs of conjugate points situated thereon. Seven arbitrary conditions, of the kind described in art. 5, are necessary and sufficient, as we shall see, to determine an exceptional correlation with singular points, or with singular lines. An exceptional correlation of the third kind, however, that is to say, one which possesses singular lines and singular points situated therein, cannot, in general, be made to satisfy more than six such conditions. Hence it is that such exceptional correlations do not present themselves in the present paper; they have been described, solely, for the sake of completeness.*

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* It is well known, and has recently been again demonstrated by Schröter, ("Journal für die reine und angewandte Mathematik," vol. 77, p. 105,) that two correlated planes can always be made to coincide in such a manner, that to each point the same line shall correspond, no matter to which of the two coincident planes that point be ascribed. The point and line, in fact, then become pole and polar relative to a conic. Now this conic degenerates to a line-pair, when the correlation has singular points; to a point-pair, when it has singular lines; and to a line-pair-point, when it has singular points and lines. Hence the connection between degenerate conics and exceptional correlations, alluded to in art. 10.

17. The following properties of the two first kinds of exceptional correlations will be frequently referred to; they are easily deduced from those already described.

a. The pole of every line, in either plane, which does not pass through the singular point of that plane, coincides with the singular point of the other plane.

b. The pole of every point, in either plane, which does not coincide with the singular point of that plane, is a determinate line passing through the singular point of the other plane.

c. The polar of every point, in either plane, not situated on a singular line of that plane, coincides with the singular line of the other plane.

d. The pole of every line, in either plane, which does not coincide with the singular line in that plane, is a determinate point in the singular line of the other plane.

Relations between the Characteristics and Singularities of any System of Correlations.

18. M_1 being an arbitrary point in one of two correlated planes, its polars, in the several correlations of any system, envelope a curve, whose class μ indicates the number of correlations of the system, in each of which the polar of M_1 passes through an arbitrary point M_2 of the second plane. But since, in each of these correlations, (and in these solely) the polar of M_2 will pass through M_1 (art. 4), μ will also be the class of the curve enveloped by the polars of an arbitrary point M_2 in the second plane.

Regarded as the number of correlations of the system, in each of which two arbitrary points M_1 and M_2 are conjugate to each other, μ may be appropriately termed the *class of the system of correlations.*

19. In like manner, the poles of an arbitrary line, in either plane, lie on a curve whose order ν may be termed *the order of the system of correlations*, since it indicates the number of correlations of the system in each of which two arbitrary lines m_1 and m_2 are conjugate to each other.

20. Between the *characteristics* μ and ν of a system of correlations, and the number of exceptional correlations which that system includes, relations exist precisely like those which have been established by Chasles between the characteristics of a system of conics satisfying four given conditions, and the number of degenerate conics included in the system.

In fact, if π denote the number of exceptional correlations with singular points, and λ the number of exceptional correlations with singular lines included in the system of correlations whose characteristics are

μ and ν, then
$$\left.\begin{aligned}\mu &= 2\nu - \pi \\ \nu &= 2\mu - \lambda\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Hence we deduce
$$\left.\begin{aligned}\lambda + \pi &= \mu + \nu \\ \lambda - \pi &= 3(\mu - \nu)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

and also
$$\left.\begin{aligned}3\mu &= 2\lambda + \pi \\ 3\nu &= \lambda + 2\pi\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

21. Since each of the relations (1) follows from the other by the Principle of Duality, a demonstration of the first will be sufficient.

If a_1 and b_1 be any two lines in the first plane, and M_2 any point in the second plane, we have to determine the number of correlations of the system (μ, ν) in each of which the polar $\overline{A_2 B_2}$ of the intersection $(a_1 b_1)$ passes through M_2. By hypothesis, there are ν correlations of the system in each of which the pole A_2 of a_1 lies in an arbitrary ray m_2 of the pencil whose centre is M_2. If the pole B_2 of b_1, in each of these ν correlations, be joined to M_2 by a ray m_2', then to each ray m_2 will correspond ν rays m_2'.

In like manner, to each ray m_2' will correspond ν rays m_2, passing, respectively, through the poles of a_1 in the several correlations in which the poles of b_1 lie in m_2'. Hence there is a (ν, ν) correspondence between the rays m_2 and m_2', and consequently there are 2ν rays, with each of which an m_2 and its corresponding m_2' coincide.

Although each of these 2ν rays passes through the poles of a_1 and b_1 in one and the same correlation, they are not all polars of $(a_1 b_1)$. For instance, if Σ_1 and Σ_2 be the singular points of one of the π exceptional correlations included in the system, the poles of the *arbitrary* lines a_1 and b_1 will coincide with Σ_2 (art. 17. **a**.), and by so doing cause m_2 and m_2' to coincide with $\overline{M_2 \Sigma_2}$. The polar of $(a_1 b_1)$ in this exceptional correlation, however, will not in general coincide with this *arbitrary* line $\overline{M_2 \Sigma_2}$, but with that ray of the pencil (Σ_2) which corresponds equianharmonically with the ray, through $(a_1 b_1)$, of the pencil (Σ_1), (arts. 16. first case, and 17. **b**.)

Exceptional correlations with singular points being, clearly, the only ones in which the poles of a_1 and b_1 coincide, it follows, as indicated by the first of the relations (1) in art. 20, that the number μ of correlations in which the polar of $(a_1 b_1)$ passes through the arbitrary point M_2 is less than 2ν by the number π of exceptional correlations in the system which possess singular points.

22. The number of exceptional correlations in a system satisfying seven given conditions being directly determined, the relations (1) of art. 20, written in the form (3), will enable us to deduce at once the characteristics of the system.

I propose to apply this method to the determination of the characteristics of the *fundamental systems*, that is to say, of those systems which satisfy elementary conditions of the following types :

A given point shall have a given polar (2 conditions).
A given line shall have a given pole (2 conditions).
Two given points shall be conjugate (1 condition).
Two given lines shall be conjugate (1 condition).

The characteristics of the fundamental systems being thus determined, it will be easy to deduce the number of correlations which satisfy any eight given elementary conditions.

All that will be then requisite in order to enable us to ascend from elementary conditions to others of a more complex character, and thus to solve the problem of the correlation of two planes in all its generality, will be a more intimate knowledge of the properties of the curves of the order ν, and of the curves of the class μ which, as we have seen (arts. 18 and 19), are associated with each system of correlations.

Enumeration and Classification of the Fundamental Systems of Correlations.

23. α, β, γ, δ being integers satisfying the condition

$$2\alpha+2\beta+\gamma+\delta=7,$$

we shall term $(\alpha\beta\gamma\delta)$ the *signature* of the system of correlations satisfying the following conditions:

α points in one plane have given polars in the other.
β right lines in the first plane have given poles in the second.
γ points and δ lines in each plane have given conjugates in the other plane.

It is obvious that the systems of correlations whose signatures are $(\alpha\beta\gamma\delta)$ and $(\beta\alpha\gamma\delta)$ are identical. The first symbol, in fact, being regarded as descriptive of the data in the one plane, the second will indicate the corresponding data in the correlated plane.

Now the total number of integral solutions of the equation

$$2\alpha+2\beta+\gamma+\delta=7$$

can be readily proved to be 52; hence *there are twenty-six distinct fundamental systems of correlations.*

24. These systems may be arranged in six groups as follows:

Group.	Signature.	
I.	(3010)	(0301)
II.	(2110)	(1201)
III.	(2030)	(0203)
	(2021)	(0212)
IV.	(1130)	(1103)
	(1121)	(1112)
V.	(1050)	(0105)
	(1041)	(0114)
	(1032)	(0123)

VI. ..... { (0070) (0007) / (0061) (0016) / (0052) (0025) / (0043) (0034)

25. In each group, the systems are arranged in two columns, such that the data of each system in one column are correlative to the data of the system opposite thereto in the other column. Now to every correlation of the system $(\alpha\,\beta\,\gamma\,\delta)$ will obviously correspond, by the Principle of Duality, a correlation of the system $(\beta\,\alpha\,\delta\,\gamma)$; moreover, the class and order of the first system will be equal, respectively, to the order and class of the second system; and for every exceptional correlation with singular points in the former there will be an exceptional correlation with singular lines in the latter, and *vice versâ*. It will suffice, therefore, to consider the systems contained in one only of the two preceding columns.

Number and Nature of Exceptional Correlations in the Fundamental Systems.

26. I proceed first to determine, directly, the number and nature of the exceptional correlations included in each of the thirteen fundamental systems whose signatures stand in the first column of the preceding article. In doing so, two associated singular points of any exceptional correlation will always be indicated by the symbols Σ_1 and Σ_2, and two associated singular lines by σ_1 and σ_2.

27. Throughout this investigation it will be assumed that, between the positions of the given points and lines, no special relation whatever exists.

(3010) 28. Let the symbol of the system (3010) be

$$\begin{pmatrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{pmatrix}.$$

$\lambda=0$ In this system there can be no exceptional correlation with singular lines. To prove this, I observe, *first*, that σ_2 could not coincide with any one of the three lines $p_2\,q_2\,r_2$, say p_2; for if it did, then Q_1 and R_1, as poles of q_2 and r_2, would lie on σ_1 (art. 17. **d**.); and A_1 not being situated thereon (art. 27), would have σ_2 for its polar, whereas by hypothesis this polar should pass through A_2; and *secondly*, that if there were a singular line σ_2 not coincident with any one of the three lines $p_2\,q_2\,r_2$, then the poles $P_1\,Q_1\,R_1$ of these lines would all lie on σ_1 (art. 17 **d**.), which is inconsistent with the assumed generality of the data by which the system is defined (art. 27).

$\pi=3$ Again, if there be a singular point Σ_1 in the system, it must coincide with one of the three points $P_1\,Q_1\,R_1$; for if it did not, $p_2\,q_2\,r_2$ would concur in Σ_2 (art. 17. **b**.), which is not the case. Now if Σ_1 were coinci-

dent with P_1, its polar, being perfectly arbitrary (art. 16.), might obviously be regarded as coincident with p_2. Moreover, q_2 and r_2, as polars of Q_1 and R_1, would then intersect in Σ_2 (art. 17. **b**.), and the polar of A_1 would be $\overline{A_2\Sigma_2}$. In this manner all the seven conditions would be fulfilled, and the exceptional correlation would be perfectly and uniquely determined (art. 16) ; the polar of any point M_1, for instance, would be the ray m_2, which passes through $(q_2 r_2)$ and satisfies the anharmonic relation $P_1(Q_1 R_1 A_1 M_1) = (q_2 r_2)(q_2 r_2 A_2 m_2)$.

In like manner, Σ_1 might coincide with Q_1, if Σ_2 were coincident with $(r_2 p_2)$; or, lastly, Σ_1 might coincide with R_1, if Σ_2 were coincident with $(p_2 q_2)$. Hence we conclude that *in the system under consideration there are but three exceptional correlations, and that each of these has singular points.* We have, consequently, by art. 20. (3), the values

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

29. The next system (2110) may be denoted by (2110)

$$\begin{pmatrix} P_1 & Q_1 & r_1 & A_1 \\ p_2 & q_2 & R_2 & A_2 \end{pmatrix}.$$

Here r_1 cannot be coincident with σ_1 ; for if it were, p_2 and q_2, as polars $\lambda=1$ of points not situated on σ_1, would be coincident with σ_2 (art. 17. **c**.) ; hence R_2 must lie on σ_2 (art. 17. **d**.) But if so, then, by art. 27, neither p_2 nor q_2 can coincide with σ_2, and as a consequence, P_1 and Q_1 must lie on σ_1. Again A_1, not being on σ_1, its polar, which by hypothesis passes through A_2, must be coincident with σ_2. Hence the *only possible correlation with singular lines is that in which these lines coincide, respectively, with* $\overline{P_1 Q_1}$ *and* $\overline{R_2 A_2}$; that this correlation satisfies the seven given conditions, and is precisely determined by them, is readily seen.

Passing to possible exceptional correlations with singular points, it $\pi=1$ is obvious that R_2 cannot coincide with Σ_2 ; for if it did, P_1 and Q_1, the poles of two lines p_2 and q_2 which do not pass through R_2, would be *coincident* in Σ_1 (art. 17. **a**.), which is not the case. Hence we infer, by art. 17. **b**., that r_1 must pass through Σ_1, and as a consequence of this, —or rather of the fact that neither P_1 nor Q_1 can, under these circumstances, coincide with Σ_1,—that p_2 and q_2 must intersect in Σ_2.

That *there is one and only one exceptional correlation which has a singular point* Σ_2 *coincident with* $(p_2 q_2)$, *and its associate* Σ_1 *situated in* r_1, and that this correlation is precisely determined by the seven given conditions, is obvious on remarking that the anharmonic relation

$$\Sigma_1(P_1 Q_1 r_1 A_1) = (p_2 q_2)(p_2 q_2 R_2 A_2)$$

is the only remaining condition which Σ_1 has to fulfil (art. 16). The point Σ_1, in fact, is the intersection, with r_1, of the line which connects A_1 with the *only* point A on $\overline{P_1 Q_1}$ which satisfies the relation

$$\overline{P_1 Q_1}(P_1 Q_1 r_1 A) = (p_2 q_2)(p_2 q_2 R_2 A_2).$$

For the system now under consideration, the equations (3) of art. 20 furnish the following values:—

$$\lambda = 1, \quad \pi = 1, \quad \mu = 1, \quad \nu = 1.$$

(2030) 30. The next system to be considered has the signature (2030), and the symbol $\begin{pmatrix} P_1 & Q_1 & A_1 & B_1 & C_1 \\ p_2 & q_2 & A_2 & B_2 & C_2 \end{pmatrix}$.

$\lambda=0$ Neither p_2 nor q_2 can here coincide with σ_2; for if either did, the polars of $A_2B_2C_2$ would be coincident with σ_1 (art. 17. **c**.), and therefore could not pass, respectively, through $A_1B_1C_1$. Hence we infer that if σ_1 exist, P_1 and Q_1 must lie on it (art. 17. **d**.). But if σ_1 were coincident with $\overline{P_1Q_1}$, the polars of $A_1B_1C_1$ would be coincident with σ_2 (art. 17. **c**.), which is inconsistent with the condition of their passing, respectively, through $A_2B_2C_2$. Hence we conclude that *there are no exceptional correlations with singular lines in the system.*

$\pi=3$ Again, if there be a singular point Σ_2, it must lie in one, at least, of the two lines p_2 and q_2, for otherwise P_1 and Q_1 would be coincident in Σ_2 (art. 17. **a**.). If Σ_2 were on p_2, but not on q_2, its associate Σ_1 would coincide with Q_1 (art. 17.), and Σ_2 itself would, by art. 16., simply have to satisfy the condition

$$Q_1 (P_1A_1B_1C_1) = \Sigma_2 (p_2A_2B_2C_2) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is clearly possible, and in one way only (art. 29).

In like manner, there is a second exceptional correlation for which Σ_1 is coincident with P_1, and Σ_2 lies on q_2 so as to satisfy the condition

$$P_1 (Q_1A_1B_1C_1) = \Sigma_2 (q_2A_2B_2C_2) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In every other exceptional correlation, Σ_2 must be coincident with (p_2q_2), and Σ_1 must satisfy the homographic relation

$$\Sigma_1 (P_1Q_1A_1B_1C_1) = (p_2q_2)\,(p_2q_2A_2B_2C_2).$$

Now it is well known that there is, in general, one and only one position for Σ_1 consistent with this relation.* It is, in fact, the fourth intersection of two conics (S_1') and (S_1'') circumscribed to the triangle $A_1B_1C_1$, one of which (S_1') passes through P_1, and is determined by the anharmonic relation

$$(S_1')\,(P_1A_1B_1C_1) = (p_2q_2)(p_2A_2B_2C_2),$$

and the other (S_1'') passes through Q_1 in such a manner that

$$(S_1'')\,(Q_1A_1B_1C_1) = (p_2q_2)\,(q_2A_2B_2C_2).$$

The only exceptional case which can present itself arises when the above two conics (S_1') (S_1'') happen to coincide. This case, however, is excluded by art. 27, since it implies the existence of a special relation between the positions of the given lines and points.

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* See Sturm's excellent paper on the *Problem der Projectivität und seine Anwendung auf die Flächen zweiten Grades*, in the "Mathematische Annalen," vol. i. p. 533, wherein many of the auxiliary theorems employed in this paper are elaborately demonstrated.

We conclude, then, that *the system under consideration contains three exceptional correlations with singular points*, and accordingly we have, as in art. 28, $\lambda = 0,\ \pi = 3,\ \mu = 1,\ \nu = 2.$

31. We proceed to the system (2021) whose symbol is (2021)

$$\begin{pmatrix} P_1 & Q_1 & A_1 & B_1 & c_1 \\ p_2 & q_2 & A_2 & B_2 & c_2 \end{pmatrix}.$$

If p_2 were coincident with a singular line σ_2, then Q_1 would lie on σ_1 $\lambda=1$ (art. 17. **d**.), and at the same time the polars of A_2 and B_2 would coincide with σ_1 (art. 17. **c**.). But, since Q_1, A_1 and B_1 are not in a line (art. 27), this is obviously inconsistent with the condition that these polars should pass, respectively, through A_1 and B_1. In like manner, q_2 cannot be coincident with σ_2; hence, if this singular line exist at all, P_1 and Q_1 must lie on its associate σ_1 (art. 17. **d**.), and the polars of A_1 and B_1 must coincide with σ_2 (art. 17. **c**.), that is to say, σ_2 must pass through A_2 and B_2.

This being clearly possible, and moreover the seven conditions being precisely sufficient to determine such an exceptional correlation,* we conclude that *the system contains one, and only one, exceptional correlation with singular lines;* the latter being coincident, respectively, with $\overline{P_1Q_1}$ and $\overline{A_2B_2}$.

Again, it can be shown, as in art. 30, that if there be a singular point $\pi=4$ Σ_2, it must lie on one, at least, of the lines $p_2 q_2$. Now if Σ_2 were situated on p_2, but not on q_2, then its associate Σ_1 would be coincident with Q_1, and the pole of c_1 would be coincident with Σ_2 (art. 17. **a**.); but as this pole must be situated on c_2, (p_2c_2) is the only possible position for Σ_2. In like manner, if Σ_2 were on q_2, but not on p_2, it would necessarily coincide with (q_2c_2), and its associate Σ_1 with P_1. Moreover, it is evident that the seven given conditions suffice precisely to determine one of each of the exceptional correlations here described.

The only remaining position possible for Σ_2 is $(p_2 q_2)$, in which case the polar of c_2, necessarily a point on c_1, would be coincident with Σ_1 (art. 17. **a**). The sole condition to be satisfied by this point Σ_1 on c_1 is

$$\Sigma_1 (P_1 Q_1 A_1 B_1) = (p_2q_2) (p_2 q_2 A_2 B_2) \ldots\ldots\ldots\ldots\ldots (1);$$

hence Σ_1 must be one of the intersections of c_1 by the conic (S_1) which passes through $P_1 Q_1 A_1 B_1$ and satisfies the anharmonic relation

$$(S_1) (P_1 Q_1 A_1 B_1) = (p_2q_2) (p_2 q_2 A_2 B_2).$$

Taking into account the two solutions of (1), we conclude that *in the system under consideration there are four exceptional correlations with singular points.*

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* The pole of any line m_1, for example, would be the point M_2, on $\overline{A_2B_2}$, determined by the relation $\overline{P_1Q_1}(P_1Q_1c_1m_1) = \overline{A_2B_2}(p_2q_2c_2M_2)$.

The equations (3) of art. 20 give us, in the present case, the values

$$\lambda = 1, \quad \pi = 4, \quad \mu = 2, \quad \nu = 3.$$

(1130) 32. We arrive next at the system (1130), which is defined by the symbol $\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & C_1 \\ p_2 & P_2 & A_2 & B_2 & C_2 \end{pmatrix}$.

$\lambda=0$ Since p_1, in this case, cannot coincide with σ_1, (for if it did, the polars of $P_1 A_1 B_1 C_1$ would be coincident with σ_2, which is obviously impossible,) σ_2, if it exist, must pass through its pole P_2, and for a like reason its associate σ_1 must pass through P_1 (art. 17. **d**.). Such an exceptional correlation, however, could not possibly satisfy the conditions in virtue of which the polars of A_1, B_1, C_1 pass, respectively, through A_2, B_2, C_2; hence we conclude that the *system under consideration includes no exceptional correlation with singular lines.*

$\pi=3$ If P_1 were a singular point, P_2, as pole of a line p_1 which does not pass through that singular point, would necessarily be its associate, and *vice versâ* (art. 17. **a**.). Such a correlation is clearly possible, and in one way only.

In any other exceptional correlation with singular points, Σ_1 would necessarily lie on p_1, since Σ_2 does not coincide with P_2; and for a like reason its associate Σ_2 would necessarily lie on p_2 (art. 17. **b**.). The sole condition to be fulfilled by these associated singular points Σ_1, Σ_2 is

$$\Sigma_1 (p_1 P_1 A_1 B_1 C_1) = \Sigma_2 (P_2 p_2 A_2 B_2 C_2) \quad \ldots\ldots\ldots\ldots\ldots (1),$$

a relation which can be satisfied in two, and only two, ways. To prove this, I observe that, as shown in art. 29, to each point S_1 on p_1 corresponds one, and only one, point S_2 on p_2 such that

$$S_1 (p_1 P_1 A_1 B_1) = S_2 (P_2 p_2 A_2 B_2);$$

whilst to this point S_2 corresponds one, and only one, point S_1' on p_1 such that $S_1' (p_1 P_1 A_1 C_1) = S_2 (P_2 p_2 A_2 C_2)$.

Hence we may say that to each point S_1 corresponds one, and only one, point S_1'; for a like reason, to each point S_1' corresponds one, and only one, point S_1. By a well-known theorem, therefore, there are two points on p_1 in each of which S_1 and S_1' coincide.

Each of these points, considered as a position of Σ_1, together with its corresponding S_2, considered as a position of Σ_2, will clearly give a solution of the relation (1). Hence we conclude that the *system under consideration includes three exceptional correlations with singular points.* For the system now under consideration, therefore, we have, as in arts. 28 and 30, the following values:

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

(1121) 33. Next, in the order of sequence adopted in art. 24, comes the system (1121), whose symbol is

$$\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & c_1 \\ p_2 & P_2 & A_2 & B_2 & c_2 \end{pmatrix}.$$

It can be shown here, precisely as in the preceding case, that if there be an exceptional correlation with singular lines, σ_1 must pass through P_1, and its associate σ_2 through P_2. But σ_1 must also pass either through A_1 or B_1; for if it did not do so, the polars of these points would coincide with σ_2 (art. 17. **c**.), which is impossible, because σ_2 could not pass at one and the same time through P_2, A_2, and B_2 (art. 27). It is obvious, however, that $\overline{P_1A_1}$ and $\overline{P_2B_2}$, as well as $\overline{P_1B_1}$ and $\overline{P_2A_2}$, will be associated singular lines of an exceptional correlation satisfying the given conditions. Hence the *system includes two, and only two, exceptional correlations with singular lines.* $\lambda=2$

Again, P_1 cannot coincide with a singular point Σ_1; for if it did, the pole of c_1 would coincide, in the associated singular point Σ_2, with P_2, the pole of p_1 (art. 17. **a**.); whereas, by hypothesis, the pole of c_1 lies on c_2. Hence Σ_2, if it exist, must lie on p_2 (art. 17. **b**.), and for a like reason its associate Σ_1 must lie on p_1. Moreover, if Σ_1 were not on c_1, its associate Σ_2 would, as pole of c_1 (art. 17. **a**.), lie on c_2; and *vice versâ*, if Σ_2 were not on c_2, its associate Σ_1 would lie on c_1. Hence we conclude that the *system includes two exceptional correlations with singular points.* In one of these Σ_1 is at (p_1c_1), and its associate Σ_2 is the sole point on p_2 which satisfies the relation $\pi=2$

$$(p_1c_1)\,(p_1P_1A_1B_1) = \Sigma_2\,(P_2p_2A_2B_2)\,;$$

in the other, Σ_2 is at (p_2c_2), and its associate Σ_1 is the sole point on p_1 which fulfils the condition

$$\Sigma_1(p_1P_1A_1B_1) = (p_2c_2)(P_2p_2A_2B_2).$$

In the present case, therefore, we have as result,

$$\lambda = 2, \quad \pi = 2, \quad \mu = 2, \quad \nu = 2.$$

34. We have now reached the fifth group of fundamental systems of correlations, the first in which is the system (1050) defined by the symbol (1050)

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & D_1 & E_1 \\ p_2 & A_2 & B_2 & C_2 & D_2 & E_2 \end{pmatrix}.$$

Here *there can be no exceptional correlations with singular lines;* for since no such line σ_1 could pass through more than two of the five given points $A_1B_1C_1D_1E_1$ (art. 27), the polars of at least three of these points would be coincident in σ_2 (art. 17. **c**.), and therefore could not all pass through the points respectively conjugate to them. $\lambda=0$

On the other hand, it is obvious here, as in art. 30, that P_1 *would be the singular point of one exceptional correlation in the system*, its associate being the sole point Σ_2 in the second plane for which $\pi=3$

$$P_1(A_1B_1C_1D_1E_1) = \Sigma_2(A_2B_2C_2D_2E_2).$$

If there be any singular point Σ_1, not coincident with P_1, then its associate Σ_2 must lie on p_2 (art. 17. **b**.), and by art. 16 the two points must satisfy the homographic relation

$$\Sigma_1(P_1A_1B_1C_1D_1E_1) = \Sigma_2(p_2A_2B_2C_2D_2E_2) \quad \ldots\ldots\ldots\ldots(1),$$

an equation of which there are two solutions. The proof of this depends upon the following Theorem:

S_2 *being any point on* p_2, *the locus of a point* S_1 *such that*

$$S_1(P_1A_1B_1C_1D_1) = S_2(p_2A_2B_2C_2D_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

is a cubic which passes through $A_1B_1C_1D_1$, *and has a double point at* P_1. In fact, it has already been proved in art. 32, that on any line drawn through one of these four points, say A_1, there are, exclusive of the point A_1 which is arbitrary, two points S_1 which satisfy the relation (2); and A_1 itself obviously does so when S_2 has the position determined, as at the end of art. 29, by the relation

$$A_1(P_1B_1C_1D_1) = S_2(p_2B_2C_2D_2).$$

Moreover, P_1 is a double point of the cubic, because, as in art. 31 (1), there are two points S_2 on p_2 such that

$$P_1(A_1B_1C_1D_1) = S_2(A_2B_2C_2D_2).$$

In like manner, the locus of a point S_1' such that

$$S_1'(P_1A_1B_1C_1E_1) = S_2(p_2A_2B_2C_2E_2)$$

is another cubic having a double point at P_1, and also passing through $A_1B_1C_1E_1$.

Now, exclusive of the double point P_1 (which counts as four intersections), and the three points A_1, B_1, C_1, none of which leads to a solution of (1), these two cubics intersect in two points Σ_1, each of which, with its associated point Σ_2 on p_2 (determined by the relation

$$\Sigma_1(P_1A_1B_1C_1) = \Sigma_2(p_2A_2B_2C_2),$$

and, therefore, the same for both cubics) obviously satisfies the relation (1). We conclude, therefore, that *the system now under consideration includes three exceptional correlations with singular points.*

We have again, therefore, the values

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

(1041) 35. The second system in the fifth group of art. 24 has the signature (1041) and the symbol

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & D_1 & e_1 \\ p_2 & A_2 & B_2 & C_2 & D_2 & e_2 \end{pmatrix}.$$

$\lambda=0$ By a process of reasoning similar to that employed in the last two articles, it may be readily shown that in the present system *there are no exceptional correlations which have singular lines.*

$\pi=6$ If any singular point Σ_1 were coincident with P_1, the pole of e_1 would coincide with its associate Σ_2 (art. 17. a.). Hence, and from art. 16, we conclude that the singular point Σ_2 associated with P_1 lies on e_2, and satisfies the relation $P_1(A_1B_1C_1D_1) = \Sigma_2(A_2B_2C_2D_2)$.

Of this there are obviously two solutions (art. 31), in other words, *there are two points* Σ_2 *on* e_2 *each of whose associates is coincident with* P_1.

Again, if there were any singular point Σ_1 not coincident with P_1, its associate Σ_2 would necessarily lie on p_2 (art. 17. **b**.), and satisfy the relation

$$\Sigma_1 (P_1 A_1 B_1 C_1 D_1) = \Sigma_2 (p_2 A_2 B_2 C_2 D_2).$$

Now all such points Σ_1, as we have proved in the preceding article, lie on a cubic S_1^3 which passes through $A_1 B_1 C_1 D_1$ and has a double point at P_1. Hence, and from the fact that Σ_2 must lie on e_2 whenever Σ_1 is not on e_1, and *vice versâ*, that Σ_1 must lie on e_1 whenever Σ_2 is not on e_2 (art. 17. **a**.), we conclude that there are *four, and only four, possible positions of* Σ_2 *on* p_2. In fact, in one of them Σ_2 coincides with $(p_2 e_2)$, and its associate Σ_1 is the sole point on S_1^3 which satisfies the relation

$$\Sigma_1 (P_1 A_1 B_1 C_1 D_1) = (p_2 e_2)(p_2 A_2 B_2 C_2 D_2),$$

whilst, in its remaining three positions on the line p_2, Σ_2 corresponds, respectively, to the three intersections $\Sigma_1' \Sigma_1'' \Sigma_1'''$ of e_1 and the cubic (S_1^3).

On the whole, therefore, *there are in the system six exceptional correlations with singular points.* We have consequently the values,

$$\lambda = 0, \quad \pi = 6, \quad \mu = 2, \quad \nu = 4.$$

36. The last of the systems we have to consider in Group V. has the signature (1032), and is defined by the symbol (1032)

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & d_1 & e_1 \\ p_2 & A_2 & B_2 & C_2 & d_2 & e_2 \end{pmatrix}.$$

If there be any singular line σ_1 it must pass through P_1, for otherwise its associate σ_2 would coincide with p_2, and the polars of $A_2 B_2 C_2$ would be coincident with σ_1 (art. 17. **c**.), instead of passing, respectively, through $A_1 B_1 C_1$. Moreover, σ_1 must likewise pass through one of the points $A_1 B_1 C_1$, otherwise the polars of these points would be coincident with σ_2, instead of passing, respectively, through $A_2 B_2 C_2$. But if σ_1 pass through P_1 and one of the points $A_1 B_1 C_1$, its associate σ_2, with which the polars of the other two will coincide (art. 17. **c**.), must pass through the conjugates of the remaining two points. On the other hand, it is obvious that all the seven given conditions could be satisfied, and in one way only, by each exceptional correlation of the above kind. $\lambda=3$

Hence we conclude that in *the system there are three exceptional correlations with singular lines;* the latter being $\overline{P_1 A_1}$ and $\overline{B_2 C_2}$, $\overline{P_1 B_1}$ and $\overline{C_2 A_2}$, and $\overline{P_1 C_1}$ and $\overline{A_2 B_2}$, respectively.

If P_1 were a singular point, $(d_2 e_2)$ would be its associate, since the poles of d_1 and e_1 would necessarily coincide with this associate (art. 17. **a**.), and, at the same time, be situated on d_2 and e_2 respectively. Moreover, it is obvious that one, and only one, exceptional correlation with these singular points satisfies the required conditions. $\pi=6$

Since, in any other possible correlation with singular points, P_1 would not coincide with Σ_1, Σ_2 would necessarily lie on p_2 (art. 17. **b**.).

Further, if Σ_2 were not coincident with either of the points in which p_2 is intersected by d_2 and e_2, then the poles of the latter lines would coincide, in $(d_1 e_1)$, with Σ_1 (art. 17. a.). This is clearly possible, and in one way only (art. 29); the necessary and sufficient relation to be satisfied by Σ_2 being $(d_1 e_1)(P_1 A_1 B_1 C_1) = \Sigma_2 (p_2 A_2 B_2 C_2)$.

If Σ_2 were coincident with $(p_2 d_2)$, then Σ_1, as pole of e_2, would be situated on e_1, and satisfy the equation

$$\Sigma_1 (P_1 A_1 B_1 C_1) = (p_2 d_2)(p_2 A_2 B_2 C_2);$$

of this equation there are obviously two solutions (art. 31).

In like manner, if Σ_2 were coincident with $(p_2 e_2)$, then there would be two positions of Σ_1, on d_2, each of which would satisfy the necessary and sufficient condition

$$\Sigma_1 (P_1 A_1 B_1 C_1) = (p_2 e_2)(p_2 A_2 B_2 C_2).$$

We conclude, therefore, that *the system under consideration contains, on the whole, six exceptional correlations with singular points.*

The following are the numerical values deducible from the above results: $\lambda = 3, \quad \pi = 6, \quad \mu = 4, \quad \nu = 5.$

(0070) 37. The first of the four systems in Group VI. has the signature (0070), and is thus defined:

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & G_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & G_2 \end{pmatrix}.$$

$\lambda = 0$ There are here, as in art. 34, *no correlations with singular lines.*

$\pi = 3$ The necessary and sufficient conditions to be fulfilled by a pair of singular points is

$$\Sigma_1 (A_1 B_1 C_1 D_1 E_1 F_1 G_1) = \Sigma_2 (A_2 B_2 C_2 D_2 E_2 F_2 G_2) \ldots\ldots (1),$$

of which equation, as Sturm and others have shown, *there are three solutions.* This important theorem results, in fact, from the following considerations: In art. 34 (1) it has been shown that, irrespective of F_2 (which is an arbitrary point), there are on any line f_2 passing through F_2, two points S_2, to each of which corresponds a point S_1, such that

$$S_1 (A_1 B_1 C_1 D_1 E_1 F_1) = S_2 (A_2 B_2 C_2 D_2 E_2 F_2) \ldots\ldots\ldots (2).$$

It is obvious, too, that F_2 itself counts as one such point; since one, and only one, point S_1 can be found to satisfy the condition

$$S_1 (A_1 B_1 C_1 D_1 E_1) = F_2 (A_2 B_2 C_2 D_2 E_2).$$

Hence, and from symmetry, we infer that the locus of the point S_2, to which another point S_1 can correspond so as to satisfy the condition (2), is a cubic passing through the six points $A_2 B_2 C_2 D_2 E_2 F_2$. The corresponding points S_1 lie, of course, on another cubic, passing through $A_1 B_1 C_1 D_1 E_1 F_1$, between whose points and those of the first cubic a (1, 1) correspondence obviously exists.

In like manner, the locus of each of the points S_1, S_2, which satisfy the relation $S_1 (A_1 B_1 C_1 D_1 E_1 G_1) = S_2 (A_2 B_2 C_2 D_2 E_2 G_2)$ ..... (3) is a cubic passing, respectively, through the six points $A_1 B_1 C_1 D_1 E_1 G_1$, and the six points $A_2 B_2 C_2 D_2 E_2 G_2$. Exclusive of the five points $A_2 B_2 C_2 D_2 E_2$ and of another point S_2, which may be termed the *satellite* of this group of five out of the seven points $A_2 B_2 C_2 D_2 E_2 F_2 G_2$, the above two cubic loci in the second plane intersect in three points Σ_2, which, with their associated points Σ_1, satisfy the relation (1).

The excluded point S_2 is that which corresponds to *every* point S_1 on the conic $(A_1 B_1 C_1 D_1 E_1)$ in such a manner that

$$S_1 (A_1 B_1 C_1 D_1 E_1) = S_2 (A_2 B_2 C_2 D_2 E_2) \text{.....(4).}$$

It is, obviously, the fourth intersection of two conics (S_2') (S_2'') circumscribed to the triangle $A_2 B_2 C_2$, one of which passes through D_2, and makes $(S_2') (A_2 B_2 C_2 D_2) = E_1 (A_1 B_1 C_1 D_1)$,

and the other passes through E_2, and makes

$$(S_2'') (A_2 B_2 C_2 E_2) = D_1 (A_1 B_1 C_1 E_1).$$

This point S_2 is excluded for the following reason: Each of the cubic loci, in the first plane, determined by (2) and (3), cuts the conic $(A_1 B_1 C_1 D_1 E_1)$ once again; but under the assumption in art. 27 these two points of intersection will not be coincident, although by (2) and (3) each will have S_2 for its correspondent. Hence, although S_2 is common to the two cubics in the second plane, it cannot be regarded as one of the points Σ_2 with which a point Σ_1 is so associated as to satisfy the relation (1). It is easy to see, however, that the satellite S_2, defined by (4), is the only one of the four intersections of the two cubics, in the second plane, which fails to lead to a solution of (1).

We conclude, therefore, that the *system under consideration contains three exceptional correlations with singular points;* accordingly, we have the values $\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$

38. We proceed to the second system (0061) of Group VI., the symbol (0061)
for which is $\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & g_2 \end{pmatrix}$.

For the reasons stated in art. 34, *this system can contain no exceptional correlations with singular lines.* $\lambda = 0$

If Σ_1, Σ_2 be the associated singular points of any exceptional correlation, the condition $\pi = 6$

$$\Sigma_1 (A_1 B_1 C_1 D_1 E_1 F_1) = \Sigma_2 (A_2 B_2 C_2 D_2 E_2 F_2) \text{.....(1)}$$

must be satisfied. Hence, as we have seen in art. 37, Σ_1 and Σ_2 must be corresponding points on two cubics; one of which passes through $A_1 B_1 C_1 D_1 E_1 F_1$, as well as through the satellite of each of the six groups of five selected from these six points, and the other passes through

$A_2\,B_2\,C_2\,D_2\,E_2\,F_2$, as well as through the six satellites similarly connected therewith.

Besides satisfying the relation (1), however, it is to be remembered that if Σ_1 be not situated on g_1, then Σ_2 must lie on g_2; and further, that if Σ_2 be not on g_2, then Σ_1 must be on g_1, whence we conclude that *in the system there are six, and only six, exceptional correlations with singular points*, and that in three of these Σ_1 is on g_1, whilst in the other three Σ_2 is on g_2.

We have thus the following system of values:

$$\lambda = 0, \quad \pi = 6, \quad \mu = 2, \quad \nu = 4.$$

(0052) 39. We pass now to the third system (0052) of the Group VI., the symbol for which is $\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & f_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & f_2 & g_2 \end{pmatrix}$.

$\lambda=0$ *Here again there are no exceptional correlations with singular lines.*

$\pi=12$ One condition to be satisfied by the singular points of every exceptional correlation which the system may contain is

$$\Sigma_1\,(A_1\,B_1\,C_1\,D_1\,E_1) = \Sigma_2\,(A_2\,B_2\,C_2\,D_2\,E_2) \ldots\ldots\ldots\ldots\ldots (1),$$

in virtue of which to each point Σ_1 will correspond, in general, one, and only one, point Σ_2, and *vice versâ*.* Hence we infer that *there can be but one position for* Σ_1 *exterior to both the lines* f_1 *and* g_1, for in such a case its associate would necessarily coincide with $(f_2 g_2)$ (art. 17. **a**.). That there will be one such position of Σ_1, however, follows from the assumption in art. 27.

In like manner, *there will be one, and only one, exceptional correlation in which* Σ_1 *will be coincident with* $(f_1 g_1)$, *and its associate exterior both to* f_2 *and* g_2.

In all other cases, Σ_1 will lie on f_1, and therefore Σ_2 on g_2; or else Σ_1 will be on g_1, and Σ_2 on f_2. To decide how often the first of these cases will present itself, it will clearly suffice to determine the order of the locus of the points Σ_2 which, by the relation (1), correspond to the several points Σ_1 of the right line f_1.

To do this we will first enquire into the number of points in which any line a_2, passing through A_2, is intersected by the required locus. Now we have already proved that the locus of the point Σ_1 which satisfies the condition

$$\Sigma_1\,(A_1\,B_1\,C_1\,D_1\,E_1) = \Sigma_2\,(a_2\,B_2\,C_2\,D_2\,E_2)$$

is a cubic passing through $B_1\,C_1\,D_1\,E_1$ and having a double point at A_1 (art. 34). Hence we may say that, exclusive of the point A_2, which has a perfectly arbitrary position on a_2, there are three positions of Σ_2 on a_2, to which correspond three positions of Σ_1 on f_1, such that the relation (1) is satisfied.

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* The only exceptions to this arise when Σ_1 (or Σ_2) coincides with one of the five points in its plane, or with the satellite S_1 (or S_2) of these five (art. 37).

It is obvious, however, that this relation will also be satisfied when Σ_2 coincides with A_2, and Σ_1 with either of the intersections of f_1 and the conic (S_1), through $B_1 C_1 D_1 E_1$, determined by the anharmonic relation

$$(S_1)\,(B_1\,C_1\,D_1\,E_1) = A_2\,(B_2\,C_2\,D_2\,E_2).$$

Hence, and from symmetry, we conclude that *if* Σ_1 *describe any right line* f_1, *the locus of the point* Σ_2, *associated therewith by the relation* (1), *is a quintic which has a double point at each of the five points* $A_2\,B_2\,C_2\,D_2\,E_2$.

This quintic, *which has moreover a sixth double point* (at the satellite S_2 of the above five points) corresponding to the intersections of f_1 and the conic $(A_1B_1C_1D_1E_1)$, cuts g_2 in five points Σ_2, to each of which corresponds, by (1), a point Σ_1 situated on f_1; whence we infer *that there are five exceptional correlations in the system under consideration, of which one singular point lies in* f_1, *and its associate in* g_2. In like manner we conclude that *there are five exceptional correlations of which one singular point lies in* g_1, *and its associate in* f_2.

Altogether, therefore, *the system includes twelve exceptional correlations with singular points, but none with singular lines.* Accordingly we have the following numerical values:

$$\lambda = 0, \quad \pi = 12, \quad \mu = 4, \quad \nu = 8.$$

40. We have now arrived at the last of the fundamental systems (0043) which need investigation. Its signature is (0043) and its symbol

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & e_1 & f_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & e_2 & f_2 & g_2 \end{pmatrix}.$$

For reasons before stated (art. 36), every singular line σ_1 must pass $\lambda=6$ through two of the four points $A_1B_1C_1D_1$, and its associate σ_2 must pass through the conjugates of the remaining two; moreover, it is easily seen that the seven given conditions can be satisfied, and that in one way only, by an exceptional correlation whose singular lines have any one of the six possible positions above indicated. Hence *in the system there are six correlations with singular lines.*

The associated singular points Σ_1, Σ_2 of every exceptional correlation $\pi=12$ in the system must, in the first place, satisfy the anharmonic relation

$$\Sigma_1\,(A_1\,B_1\,C_1\,D_1) = \Sigma_2\,(A_2\,B_2\,C_2\,D_2).$$

The position of one of the two points being known, therefore, we have at once a conic on which the other must be situated.

Now the point Σ_1 must lie in one, at least, of the three lines $e_1 f_1 g_1$, otherwise the poles of these lines would be coincident in Σ_2 (art. 17. **a.**), which is obviously inconsistent with the condition of their being situated, respectively, on e_2, f_2, and g_2. Again, if Σ_1 lie on one only of the three lines $e_1 f_1 g_1$, then its associate will necessarily coincide with the intersection of the conjugates of the other two, and Σ_1 itself will be

coincident with one of the two points in which the first line is intersected by a conic determined by (1). Lastly, if Σ_1 coincide with an intersection of any two of the lines $e_1 f_1 g_1$, then Σ_2 will necessarily coincide with one of the two points in which the conjugate of the third line is intersected by another conic, likewise determined by (1). In every possible case, therefore, one of the associated points Σ_1, Σ_2 must coincide with the intersection of two of three given lines, and the other must have one of two known positions on the conjugate of the third.

Taking all possible combinations into consideration, and remembering that each one of them leads to two exceptional correlations satisfying the seven conditions, we conclude that *the system under consideration contains twelve exceptional correlations with singular points.* We have consequently the following system of values:

$$\lambda = 6, \quad \pi = 12, \quad \mu = 8, \quad \nu = 10.$$

41. The class, order, and singularities of each of the thirteen fundamental systems, arranged in the left-hand column in art. 24, having now been determined, those of the remaining thirteen systems may, as stated in art. 25, be deduced by the Principle of Duality. The results may be thus tabulated:

Group.	Signature.	Characteristics.		Singularities.	
	$(\alpha\beta\gamma\delta)$	μ	ν	π	λ
I.	(3010)	1	2	3	0
	(0301)	2	1	0	3
II.	(2110)	1	1	1	1
	(1201)	1	1	1	1
III.	(2030)	1	2	3	0
	(2021)	2	3	4	1
	(0212)	3	2	1	4
	(0203)	2	1	0	3
IV.	(1130)	1	2	3	0
	(1121)	2	2	2	2
	(1112)	2	2	2	2
	(1103)	2	1	0	3
V.	(1050)	1	2	3	0
	(1041)	2	4	6	0
	(1032)	4	5	6	3
	(0123)	5	4	3	6
	(0114)	4	2	0	6
	(0105)	2	1	0	3

Group.	Signature. $(\alpha\beta\gamma\delta)$	Characteristics. μ	ν	Singularities. π	λ
VI.	(0070)	1	2	3	0
	(0061)	2	4	6	0
	(0052)	4	8	12	0
	(0043)	8	10	12	6
	(0034)	10	8	6	12
	(0025)	8	4	0	12
	(0016)	4	2	0	6
	(0007)	2	1	0	3*

Number of Correlations satisfying eight Elementary Conditions.

42. The number of correlations which satisfy any eight elementary conditions may be readily determined from the preceding Table. In fact, if we indicate this number by the symbol $[\alpha\beta\gamma\delta]$, where $\alpha, \beta, \gamma, \delta$ have the same signification as before (art. 23), but now satisfy the equation $2\alpha+2\beta+\gamma+\delta = 8$,

we shall clearly have $[\alpha\beta\gamma\delta]$ *equal to the class of the system of correlations* $(\alpha\beta\overline{\gamma-1}\,\delta)$, *as well as to the order of the system of correlations* $(\alpha\beta\gamma\overline{\delta-1})$. But the systems $(\alpha\beta\gamma\delta)$ and $(\beta\alpha\gamma\delta)$ being identical (art. 23), and the *class* and *order* of the system $(\alpha\beta\gamma\delta)$ being the same, respectively, as the *order* and *class* of the system $(\beta\alpha\delta\gamma)$ (art. 25), it follows that $[\alpha\beta\gamma\delta] = [\beta\alpha\gamma\delta] = [\alpha\beta\delta\gamma] = [\beta\alpha\delta\gamma]$,

in other words, that γ and δ as well as α and β are interchangeable in the symbol $[\alpha\beta\gamma\delta]$. The following Table, therefore, gives the number

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* The meanings of the symbols being suitably modified, the results contained in this Table are at once applicable to the case of two homographic planes. For by the method employed in art. 15, we have, corresponding to every system of correlations $(\alpha\beta\gamma\delta)$ in two planes Π_1, Π_2, *a system of homographic relations* $(\alpha\beta\gamma\delta)$ in the planes Π_1, Π', which satisfy seven conditions of the following type: α points in the first plane correspond to given points in the second; β lines in the first plane correspond to given lines in the second; γ points in the first plane each correspond to a point on a given line in the second; and δ lines in the first plane each correspond to a line through a given point in the second. Every such system of homographic relations contains exceptional ones. In π of these there is a singular point in the first plane and a singular line in the second, whilst in λ others there is a singular line in the first plane and a singular point in the second (art. 14). The points which correspond, in the several homographic relations of a system, to a given point in the first plane, lie on a curve of the order μ in the second; those which correspond to a point in the second plane, however, lie on a curve of the order ν in the first. Similarly, the lines which correspond to a given line in the first plane envelope a curve of the class ν in the second, whilst those which correspond to a line in the second plane envelope a curve of the class μ in the first.

It is worth observing that if each point in the plane Π_1 be connected with every point in the plane Π' which corresponds thereto, in the several homographic relations of a system whose characteristics are μ and ν, we shall have, in space, a *complex of the degree* $\mu+\nu = \lambda+\pi$ [art. 20 (2)], of which Π_1 and Π' are singular planes. We have consequently twenty-six distinct complexes associated with the several fundamental systems of homographic relations satisfying seven conditions.

of solutions in all possible cases where the eight given conditions are of the elementary kind described in art. 22 :

$$[4000] = 1;$$
$$[3100] = 1;$$
$$[2200] = 0;$$
$$[3020] = 1,\quad [3011] = 2;$$
$$[2120] = 1,\quad [2111] = 1;$$
$$[2040] = 1,\quad [2031] = 2,\quad [2022] = 3;$$
$$[1140] = 1,\quad [1131] = 2,\quad [1122] = 2;$$
$$[1060] = 1,\quad [1051] = 2,\quad [1042] = 4,\quad [1033] = 5;$$
$$[0080] = 1,\quad [0071] = 2,\quad [0062] = 4,\quad [0053] = 8,\quad [0044] = 10.*$$

43. The only results in the above Table which cannot be obtained in the manner described in art. 42 are the first three. Of these, however, the first is well known, and has already been alluded to in art. 3. The second differs in form only from the first ; for whenever the poles of three lines are given, as in the first case, the polars of three points (the intersections of the given lines) are always known, and *vice versâ*. With respect to the third result, it will be at once seen, on writing the eight conditions in full thus :

$$\begin{matrix} P_1, & Q_1, & r_1, & s_1 \\ p_2, & q_2, & R_2, & S_2 \end{matrix}$$

that they cannot possibly be satisfied by any correlation unless the anharmonic ratios $(r_1 s_1)(P_1 Q_1 r_1 s_1)$ and $\overline{R_2 S_2}(p_2 q_2 R_2 S_2)$ are equal to one another. The assumption of any such equality, however, would be inconsistent with that already made in art. 27, in virtue of which the positions of the given points and lines are perfectly arbitrary.

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* This Table obviously gives, also, the number of ways in which two planes may be rendered homographic (or put in perspective with each other), so as to satisfy eight conditions of the kind described in the note to art. 41.

Connexes determined by Fundamental Systems of Correlations.

44. In art. 18 allusion was made to the series (doubly infinite) of curves of the class μ, and to the series of curves of the order ν, which every system of correlations, satisfying seven conditions, determines in each of the two planes; and in art. 22 it was observed that upon a more intimate knowledge of the properties of these curves must depend the solution of the general problem of correlation. Although unprepared, at present, to treat this wide subject exhaustively, I propose, before terminating the present paper, to indicate briefly the more salient features of a few of the simplest series of curves of the above kind.

45. Before commencing to do so, I may observe that, according to the terminology employed by Clebsch in a very suggestive posthumous paper recently published in the "Mathematische Annalen" (vol. 6, p. 203), each point of one of our two planes, and its polar in any correlation of a system, constitute an *element of a connex* of the class μ and order ν;* whilst each point in the other plane, and any one of its polars constitute an element of the *conjugate connex*. As illustrations, therefore, of the Theory originated by the eminent geometer, whose loss to science is so universally deplored, as well as on account of their application to the general problem of correlation, the well-defined conjugate connexes to which the several fundamental systems of correlations lead are well worthy of full investigation.†

46. Each curve of the class μ may be termed the *representative* of the point by whose polars it is enveloped, and each curve of the order ν the *representative* of the line of whose poles it is the locus. It is obvious that *every representative of a point touches all the singular lines in its plane, and every representative of a line passes through all the singular points in its plane.*

47. Each of the systems (3010), (2030), (1130), (1050), and (0070) leads to a pair of conjugate connexes of the first class and second order. In each of these five systems, a point in either plane is represented by a point in the other plane, (in other words, every point of the plane has its conjugate, exactly in the same manner as the given points have,) whilst every line is represented by a conic passing through the three fixed singular points of the three exceptional correlations which each system includes. Connex (12)

* A connex $(\mu\nu)$, of the class μ and order ν, is defined by a given relation $(\xi, \eta, \zeta)^{\mu}(x, y, z)^{\nu} = 0$ between the coordinates x, y, z of a point, and the coordinates ξ, η, ζ of an associated right line.

† I may here mention that this investigation was in a tolerably advanced state eighteen months ago, when my studies were interrupted by other duties. Uncertainty as to when these studies may be resumed, has at length induced me, contrary to my original intention, to publish the present paper, before completing the whole enquiry.

We have here, in fact, the well-known case of the *Quadric Correspondence* of two planes, presented in its most general form; the three pairs of singular points being identical with the three pairs of Principal Points. The five systems above enumerated correspond, obviously, to the five different ways of determining a Quadric Correspondence by means of given Corresponding Points, Principal Points, and Principal Lines.*

Connex (21) 48. The systems (0301), (0203), (1103), (0105), and (0007) all lead to connexes of the second class and first order. Every right line, in each of these systems, is represented by a right line; in other words, every line has its conjugate, in exactly the same sense as the given lines have. Every point, however, is represented by a conic touching the singular lines of the three exceptional correlations which each system includes. In short, we have here a correspondence established between two planes which is, simply, the correlative of the ordinary Quadric Correspondence.†

Connex (11) 49. Of the remaining sixteen fundamental systems of correlations, by far the simplest are the two whose signatures are (2110) and (1201), both of which lead to one and the same kind of connex of the first class and first order.

In illustration of this I will consider briefly the first of the two systems. Its complete definition is given by the symbol

$$\begin{pmatrix} \mathrm{P}_1 & \mathrm{Q}_1 & r_1 & \mathrm{A}_1 \\ p_2 & q_2 & \mathrm{R}_2 & \mathrm{A}_2 \end{pmatrix}$$

and, as shown in art. 29, it contains two exceptional correlations; of which one has the singular lines $\sigma_1 \equiv \overline{\mathrm{P}_1\,\mathrm{Q}_1}$ and $\sigma_2 \equiv \mathrm{R}_2\overline{\mathrm{A}_2}$, whilst the other has the singular points $\Sigma_2 \equiv (p_2 q_2)$, and Σ_1; the latter point being so situated, on r_1, that

$$\Sigma_1\,(\mathrm{P}_1\,\mathrm{Q}_1\,r_1\,\mathrm{A}_1) = \Sigma_2\,(p_2\,q_2\,\mathrm{R}_2\,\mathrm{A}_2).$$

Since the characteristics are $\mu=1$ and $\nu=1$, it is obvious, *first*, that the polars of any point (say M_1) in any *two* correlations of the system (*e.g.*, in the two exceptional ones) determine, by their intersection, the point M_2, through which all the other polars of M_1 pass; and, *secondly*, that the poles of any right line m_1 in the two exceptional correlations determine the line m_2, upon which all the other poles of m_1 are situated. Now if M_1 be neither coincident with Σ_1 nor situated on σ_1, its polar in the exceptional correlation with singular lines will coincide with the

* See Reye on *Geometrische Verwandschaften zweiten Grades*, in the *Zeitschrift für Mathematik und Physik*, vol. xi., 1865.

† The fact that we have no distinctive name for this correspondence is doubtless due to the circumstance that no terms correlative to the very convenient ones *quadric*, *cubic*, &c. are yet in general use. A special case of the correspondence in question was briefly described in my paper "On the Quadric Inversion of Plane Curves," published in the Proceedings of the Royal Society, vol. xiv., 1865, and in the "Annali di Matematica pura ed applicata," tom. vii., Roma, 1865. It was also distinctly referred to in the paper by Reye above cited.

singular line σ_2 (17. **a**.), and its polar m_2, in the exceptional correlation with singular points, will pass through Σ_2 (17. **b**.), and be determined by the relation $\Sigma_1 (P_1 Q_1 r_1 M_1) = \Sigma_2 (p_2 q_2 R_2 m_2)$.

Consequently, every such point M_1 is *represented*, in the connex, by a *point* $M_2 \equiv (m_2 \sigma_2)$ *on the singular line* σ_2. If M_1 were coincident with Σ_1, however, m_2 would be perfectly indeterminate (art. 16.), and its representative M_2 would, as a consequence, be *an indeterminate point on* σ_2. Lastly, if M_1 were situated on σ_1, then, although m_2 would be determined by (1), as before, the polar of M_1 in the exceptional correlation with singular lines would be an indeterminate line through $(m_2 \sigma_2)$ (art. 16.); and since the latter in one of its possible positions would be coincident with m_2, the representative of M_1 must be regarded as an *indeterminate point in* m_2.*

50. The peculiar *point by point* representation to which we are led by the system of correlations under consideration is such, therefore, that 1) the singular point of either plane is represented by an indeterminate point in the singular line of the other plane; 2) a point in the singular line of either plane is represented by an indeterminate point in a determinate line through the singular point of the other plane; 3) every other point in the first plane is represented by a perfectly determinate point in the singular line of the second plane. From similar considerations we infer that the *line by line* representation, to which the present system of correlations leads, may be briefly described thus: 1) a singular line is represented by an indeterminate line through a singular point; 2) a line through a singular point, by an indeterminate line through a determinate point on a singular line; 3) every other line, in either plane, is represented by a perfectly determinate line through a singular point.

51. It will be observed that the representation just described is not, even in an exceptional sense, homographic; since one of the characteristic properties of homographic correspondence is not fulfilled by it,—I mean that property, in virtue of which to a point in a line always corresponds a point in the corresponding line. The representation above indicated, however, as well as that termed homographic, is to be regarded as one of the forms to which a connex of the first class and first order may lead.

52. It is also worth observing that in the connex now under consideration the locus of the points which represent those situated in a given right line of either plane, breaks up into the representative of that line in the other plane, and the singular line of the latter plane;

* It is obvious that, in all the correlations of the system, except that which has singular lines, the polars of a point on a singular line in either plane are *coincident* with a line which passes through the singular point of the other plane.

and further, that the envelope of the lines which represent those passing through a given point of either plane, breaks up into the representative of that point in the other plane, and the singular point of the latter plane.

We have here, in fact, the simplest case of a general theorem which holds for all the connexes determined by the fundamental systems of art. 41, and which may be thus enumerated:

*The curve, of the order ν, which represents any straight line, is always a constituent part of the envelope of the curves, of the class μ, which represent the several points of that line; and, vice versâ, the curve, of the class μ, which represents any point, is always a constituent part of the envelope of the curves, of the order ν, which represent the several right lines passing through that point.** In proof of this, it will be sufficient to show that if A_1 and a_2 be pole and polar in any correlation of a system, then, in that same correlation, the tangent at A_1 to the representative of a_2, and the point of contact of a_2 with the representative of A_1, will also be pole and polar.† Now in the correlation immediately following the *first*, let a'_2 be the polar of A_1, and A'_1 the pole of a_2, in which case, of course, $(a_2\, a'_2)$ will be the pole of $\overline{A_1 A'_1}$. Then, ultimately, as the latter correlation approaches to identity with the first, $(a_2\, a'_2)$ becomes the point of contact of a_2 with the representative of A_1, and $\overline{A_1 A'_1}$ becomes the tangent at A_1 to the representative of a_2.

Connex (22)

53. The connexes, of the second class and second order, to which the systems (1121) and (1112) give rise, are essentially the same in character. We may confine our attention, therefore, to the first system, which has been denoted in art. 33 by the symbol

$$\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & c_1 \\ p_2 & P_2 & A_2 & B_2 & c_2 \end{pmatrix},$$

and found to include four exceptional correlations. One of these has the singular lines $\sigma'_1 \equiv \overline{P_1 A_1}$, $\sigma'_2 \equiv \overline{P_2 B_2}$; another the singular lines $\sigma''_1 \equiv \overline{P_1 B_1}$ and $\sigma''_2 \equiv \overline{P_2 A_2}$; a third has the singular points $\Sigma'_1 \equiv (p_1 c_1)$ and Σ'_2, the latter being so situated on p_2 that

$$(p_1 c_1)\,(p_1 P_1 A_1 B_1) = \Sigma'_2\,(P_2 p_2 A_2 B_2);$$

and the fourth has the singular points Σ''_1 and $\Sigma''_2 \equiv (p_2 c_2)$, of which the former is so situated on p_1 that

$$\Sigma''_1\,(p_1 P_1 A_1 B_1) = (p_2 c_2)\,(P_2 p_2 A_2 B_2).$$

* In studying the properties of the various connexes, this theorem is of the greatest utility. It was to it, as a fruitful source of theorems on Envelopes, that I referred in a Note published in the *Educational Times* for December, 1871 (Reprint, vol. xvi., p. 99). The theorems demonstrated in that Note were special cases of more general ones, deducible from the connex treated of in art. 53.

† It is in consequence of this that the two connexes, to which every system of correlations leads, are *conjugate* in the manner defined by Clebsch, at p. 208 of the paper referred to in art. 45.

The conjugate connexes to which we are led by the present system are such that every point, in either plane, is represented by a conic which touches the two singular lines of the other plane, and every right line in the first plane is represented by a conic which passes through the two singular points of the second.

54. It should be observed, however, that *the conics which represent the several points of each singular line, degenerate to point-pairs.*

Thus, if M_1 be a point on the singular line σ'_1, its polar, in the exceptional correlation whose singular lines are σ'_1 and σ'_2, is an *indeterminate line* through the point M'_2, on σ'_2, which is determined by the relation

$$\sigma'_1 (P_1 p_1 c_1 M_1) = \sigma'_2 (p_2 P_2 c_2 M'_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence we at once infer that the conic which represents M_1 breaks up into the point M'_1, and another point M_2. The latter point must be situated on σ''_2, because this line is the polar of M_1 in the exceptional correlation whose singular lines are σ''_1 and σ''_2 (17. **c**.), and it must also satisfy the relation

$$\sigma'_1 (P_1 p_1 A_1 M_1) = \sigma''_2 (p_2 P_2 A_2 M_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

because the polars of M_1, in the correlation whose singular points are Σ'_1 and Σ'_2, as well as its polar in the correlation whose singular points are Σ''_1 and Σ''_2, obviously pass through the point M_2 thus determined. This point M_2, it will be observed, is conjugate to M_1 in precisely the same sense that A_2 is conjugate to A_1; in fact, we may say that *every point on a singular line of either plane has a conjugate point situated on the non-associated singular line of the other plane.*

The relations (1) and (2) also show that there is a (1, 1) correspondence between the points M_2 and M'_2 of a nature such that when one of them coincides with P_2, or falls upon p_2, the other does the same. Hence, by a well known theorem, we conclude that $\overline{M_2 M'_2}$ always passes through a fixed point S'_2 on p_2. In like manner, it can be shown that every point N_1 on the singular line σ''_1 is represented by a point-pair, N_2 (on σ'_2) and N''_2 (on σ''_2), whose double tangent $\overline{N_2 N''_2}$ passes through a fixed point S''_2 on p_2. It can be shown, moreover, that *the six points* Σ'_2, Σ''_2, M_2, M'_2, N_2, N''_2 *always lie on a conic;* and by the theorem of art. 52 we at once infer that *this is the conic which represents the arbitrary line* $\overline{M_1 N_1}$.

55. In a precisely similar manner, the following properties may be established:—

Every line through a singular point of either plane has a conjugate line passing through the non-associated singular point of the other plane.

Every right line m_1 *passing through the singular point* Σ'_1 *is represented by a line-pair* m_2, m'_2, *of which* m'_2 *passes through* Σ'_2 *and* m_2 *through*

Σ_2'', and the double point $(m_2 m_2')$ of this line-pair lies on a fixed line s_2' passing through P_2.

Every line n_1 passing through Σ_1'' is represented by a line-pair $n_2 n_2''$, of which n_2'' passes through Σ_2'', and n_2 through Σ_2'; and the double point $(n_2 n_2'')$ of this line-pair lies on a fixed line s_2'' passing through P_2.

Every four such lines m_2, m_2', n_2, n_2'', together with the singular lines σ_2', σ_2'' constitute six tangents of the conic which represents the arbitrary point $(m_1 n_1)$.

56. The degenerate conics of the first plane which represent points and lines in the second, have precisely similar properties, that is to say,—

The points of the point-pairs lie on the singular lines σ_1', σ_1'', whilst their double tangents envelope a point-pair S_1', S_1'', *whose own double tangent is coincident with the line p_1. The lines of the line-pairs pass through the singular points Σ_1', Σ_1'', whilst their double points describe a line-pair $s_1' s_1''$, whose own double point is coincident with* P_1.

57. A degeneration such as that above described occurs in every fundamental system of correlations. We may in fact say, generally:

The curve of the class μ which represents any point through which pass κ singular lines, degenerates to κ points, situated on the associated singular lines, and to a curve of the class $\mu - \kappa$, which touches the remaining $\lambda - \kappa$ singular lines.

The curve of the order ν which represents any line in which lie κ singular points, degenerates to κ lines through the associated singular points, and to a curve of the order $\nu - \kappa$ which passes through the remaining $\pi - \kappa$ singular points.

58. Next in point of simplicity are the connexes determined by the systems of correlations of which (2021) and (2012) are the signatures. The first of these connexes is of the second class and third order, the second of the third class and second order.

Connex (23) The properties of the latter being at once deducible from those of the former by the Principle of Duality, we may confine our attention to the first of the above systems, the symbol for which is

$$\begin{pmatrix} P_1 & Q_1 & A_1 & B_1 & c_1 \\ p_2 & q_2 & A_2 & B_2 & c_2 \end{pmatrix}.$$

In art. 31 the existence was proved of a pair of singular lines $\overline{P_1 Q_1}$ and $\overline{A_2 B_2}$, and four pairs of singular points, viz., P_1 and $(q_2 c_2)$, Q_1 and $(p_2 c_2)$, Σ_1' and $(p_2 q_2)$, and Σ_1'' and $(p_2 q_2)$, where Σ_1', Σ_1'' are the intersections, with c_1, of the conic (S_1) which passes through $P_1 Q_1 A_1 B_1$ so as to make the anharmonic ratio of these four points on it equal to

$$(p_2 q_2)\,(p_2 q_2 A_2 B_2).$$

59. The mode in which the conics and cubics representing special

points and lines degenerate, is very instructive. Instead of attempting an exhaustive discussion of the question, however, I must limit myself here to those cases of degeneration which most elucidate the general character of the representative curves.

Every point C_2 on the singular line $\overline{A_2B_2}$ is represented, as we have already stated in art. 57, by a point-pair C_1, C_1'. The point C_1' is so situated on the associated singular line $\overline{P_1Q_1}$ as to satisfy the condition

$$\overline{P_1Q_1}\,(P_1Q_1c_1C_1') = \overline{A_2B_2}\,(p_2q_2c_2C_2) \quad \ldots\ldots\ldots\ldots\ldots (1);$$

the polar of C_2, in the exceptional correlation whose singular lines are $\overline{P_1Q_1}$ and $\overline{A_2B_2}$ being an *indeterminate* line through C_1'. The point C_1, through which the polar of C_2 in every other correlation of the system passes, is situated on the conic (S_1) and fulfils the condition

$$(S_1)(P_1\,Q_1\,A_1\,B_1\,C_1) = (p_2\,q_2)(P_2\,q_2\,A_2\,B_2\,C_2) \quad \ldots\ldots\ (2),$$

as is at once obvious on considering the polars of C_2 in the two correlations whose singular points are Σ_1' and (p_2q_2), and Σ_1'' and (p_2q_2).

60. Between the points of the conic (S_1), and those of the line $\overline{A_2B_2}$, therefore, a (1, 1) correspondence is established, such that any two corresponding points $C_1\,C_2$ thereof are conjugate to each other in the same sense as A_1A_2 and B_1B_2 are by hypothesis. The system of correl. tions, in fact, would suffer no change if C_1C_2 were substituted in place of either of the latter pairs of conjugate points.

61. *The conic*, in the second plane, *which represents any arbitrary point*, say A_1, *of the conic* (S_1), *is a line-pair whose double point coincides with the corresponding point* A_2 *on* A_2B_2. This follows from the circumstance that, in the system now under consideration, there are *two* correlations for which the polar of A_1 coincides with an arbitrary line a_2 passing through A_2; in other words, there are two correlations which satisfy the eight conditions

$$\begin{matrix} P_1,\ Q_1,\ A_1,\ B_1,\ c_1 \\ p_2,\ q_2,\ a_2,\ B_2,\ c_2 \end{matrix}$$

In proof of this we need merely refer to the Table in art. 42, where it will be seen that $[3011] = 2$.

62. From the above we infer that *every line* x_2 *in the second plane is represented by a cubic* x_1^3 *which has a double point* C_1 *on the conic* (S_1). For if C_1 be the point on (S_1) which corresponds, by art. 60, to the point C_2 in which $\overline{A_2B_2}$ is intersected by x_2, then, as we have just seen, there are two correlations of the system for which the pole of x_2 is coincident with C_1. *The cubic* x_1^3 *obviously passes also through the singular points* P_1, Q_1, Σ_1', Σ_1'' *as well as through the point* C_1' *on* $\overline{P_1Q_1}$, *which*, by (1) of art. 59, *corresponds to* C_2 *on* $\overline{A_2B_2}$.

63. Lastly, since two of the singular points of the second plane coin-

cide in $(p_2 q_2)$ it is obvious that *every line* x_1 *in the first plane is represented by a cubic* x_2^3, *which has a double point at* $(p_2 q_2)$, *and passes, likewise, through the singular points* $(p_2 c_2)$, $(q_2 c_2)$, *as well as through the point* C_2, *on* $\overline{A_2 B_2}$, *which corresponds*, by (1) of art. 59, *to the point* C_1' *in which* x_1 *intersects* $\overline{P_1 Q_1}$, *and through the points* D_2 *and* E_2, on $\overline{A_2 B_2}$, *which correspond*, by (2) of art. 59, *to the points* D_1 *and* E_1 *in which* x_1 *intersects the conic* (S_1).

64. The remark in art. 61 is susceptible of generalisation in the form of the following useful Theorems, with the enunciation of which I will conclude the present paper:

Theorem (i.):—*In a system of correlations* $(\alpha\beta\gamma\delta)$, *the curve, of the class* $[\alpha\beta(\gamma+1)\delta]$ (art. 42), *which represents either of two conjugate points* A_1, A_2, *breaks up into the other, together with a point on each of the singular lines associated with those which pass through the former. The multiplicity of* A_2 *on the representative of* A_1 *is* $[(\alpha+1)\beta(\gamma-1)\delta]$, *and that of* A_1 *on the representative of* A_2 *is* $[\alpha(\beta+1)(\gamma-1)\delta]$. *The number of singular lines which pass through* A_1 *is*

$$[\alpha\beta(\gamma+1)\delta]-[(\alpha+1)\beta(\gamma-1)\delta],$$

and the number of those which pass through A_2 *is*

$$[\alpha\beta(\gamma+1)\delta]-[\alpha(\beta+1)(\gamma-1)\delta].$$

Theorem (ii.):—*In a system of correlations whose signature is* $(\alpha\beta\gamma\delta)$ *the curve, of the order* $[\alpha\beta\gamma(\delta+1)]$ (art. 42), *which represents either of two conjugate lines* a_1, a_2, *breaks up into the other, together with a line through each of the singular points associated with those situated on the former. The multiplicity of* a_2 *on the representative of* a_1 *is* $[\alpha(\beta+1)\gamma(\delta-1)]$, *and that of* a_1 *on the representative of* a_2 *is* $[(\alpha+1)\beta\gamma(\delta-1)]$. *The number of singular points situated on* a_1 *is*

$$[\alpha\beta\gamma(\delta+1)]-[\alpha(\beta+1)\gamma(\delta-1)],$$

and the number of those situated on a_2 *is*

$$[\alpha\beta\gamma(\delta+1)]-[(\alpha+1)\beta\gamma(\delta-1)].$$

On the Contact of Quadrics with other Surfaces. By William Spottiswoode, M.A., F.R.S.

[*Read May 14th*, 1874.]

Let x, y, z, t; x_1, y_1, z_1, t_1; ..... be the coordinates of the points P, P_1, ...; and let ξ, η, ζ, θ be the current coordinates. Also let

$$U = (\xi, \eta, \zeta, \theta)^n = 0, \quad V = (\xi, \eta, \zeta, \theta)^2 = 0 \ldots\ldots\ldots\ldots (1)$$

be the equations to the surface U, and the quadric V whose contact with U is the subject of consideration. Further, let us write symbolically

$$\left.\begin{array}{l}(x, y, z, t)^n = 0^n, \quad (x_1, y_1, z_1, t_1)^n = 1^n, \ \ldots\ldots \\ (x, y, z, t)^{n-1}(x_1, y_1, z_1, t_1) = 0^{n-1}1, \ \ldots\ldots\end{array}\right\} \ldots\ldots\ldots (2).$$

Then the condition that the points P, P_1, ... may lie on the surface U will be

$$0^n = 0, \quad 1^n = 0, \ \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Further, if V touch U at any point P, we have, as usual,

$$\frac{\delta_x U}{\delta_x V} = \frac{\delta_y U}{\delta_y V} = \frac{\delta_z U}{\delta_z V} = \frac{\delta_t U}{\delta_t V} \ \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

But by multiplying numerators and denominators of these ratios first by x, y, z, t, and adding; then by x_1, y_1, z_1, t_1, and adding; and so on, we may replace the equations (4) by the following:

$$\frac{0^{n-1}1}{01} = \frac{0^{n-1}2}{02} = \frac{0^{n-1}3}{03} \ \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

P_1, P_2, P_3 being any points whatever. Conversely, if we multiply numerators and denominators of (5) by the determinants of the matrix

$$\left\|\begin{array}{cccc} x, & x_1, & x_2, & x_3 \\ y, & y_1, & y_2, & y_3 \\ z, & z_1, & z_2, & z_3 \end{array}\right\|$$

taken in cyclic order, and add in the usual way, we shall find the result $= \frac{\delta_t U}{\delta_t V}$; and similarly, by employing matrices whereof the lines consist of y's, z's, t's; z's, x's, t's; x's, y's, t's, we should reproduce the other ratios of the system (4).

If V touch U in other points beside V, say in P_1, P_2, ..., we may form the system

$$\left.\begin{array}{ccccc} . & \frac{0^{n-1}1}{01} & = & \frac{0^{n-1}2}{02} & = \ldots \\ \frac{1^{n-1}0}{10} & = \ . & = & \frac{1^{n-1}2}{12} & = \ldots \\ \frac{2^{n-1}0}{20} & = & \frac{2^{n-1}1}{21} & = \ . & = \ldots \\ \ldots & \ldots & & . & \ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

and from every three such sets of equations we may eliminate the factors belonging to the quadric (*i. e.*, the factors 12, 20, 01, ...), and obtain an equation involving only terms belonging to U, viz.,

$$0^{n-1}1 \,.\, 1^{n-1}2 \,.\, 2^{n-1}0 = 0^{n-1}2 \,.\, 2^{n-1}1 \,.\, 1^{n-1}0 \ \ldots\ldots\ldots\ldots (7),$$

say $(0, 1, 2) = 0$. This is therefore a relation which must hold good for any three points on the surface U, in order that it may be possible to draw a quadric touching it at those points. Hence, if two of the

points, say P, P_1, be taken arbitrarily, the third, P_2, must lie on the surface (7); it will consequently lie on the curve $2^n = 0$, $(0, 1, 2) = 0$. The degree of (7) is n.

If V touches U at a fourth point P_3, we shall have an additional condition. In fact, if we take the four points three and three together, we shall arrive at four conditions which, like (7), may be thus represented:

$$(1, 2, 3) = 0, \quad (0, 2, 3) = 0, \quad (0, 1, 3) = 0, \quad (0, 1, 2) = 0 \quad \ldots (8).$$

But these four are equivalent to only two independent conditions, as may either be seen *à priori* from the system (6), or may be verified *à posteriori*. If therefore we select P_2 arbitrarily on the curve $2^n = 0$, $(0, 1, 2) = 0$; P_3 will lie upon one of the points of intersection of the surface $(0, 2, 3) = 0$ with the curve in question.

There is another way in which the equation $(0, 1, 2) = 0$ may be regarded. If the points P, P_1, P_2 be chosen arbitrarily, the equation in question will express a relation which must subsist between the coefficients of U in order that the contact may be possible. In that case the coefficients will be subject to four conditions; viz., three in order that the surface may pass through P, P_1, P_3, *i. e.*, $0^n = 0$, $1^n = 0$, $2^n = 0$; and one for the contact, viz., $(0, 1, 2) = 0$. Similarly, in order that contact may be possible at four arbitrary points, we should have two additional conditions, viz., $3^n = 0$, $(0, 1, 3) = 0$. And generally, in order that contact may be possible at m points, we should have $2(m-1)$ conditions. The greatest number of points at which contact can be rendered possible will be found by making the number of conditions equal to the number of independent constants in the expression for U. Thus, if U has its full number of constants, we should have

$$2(m-1) = \frac{(n+1)(n+2)(n+3)}{1.2.3} - 1,$$

i. e.,

$$m \leqq \frac{1}{2}\frac{(n+1)(n+2)(n+3)}{1.2.3} + \frac{1}{2}.$$

But it must be noticed that this number cannot in general be attained. For the greatest number of arbitrary points through which a quadric can be drawn is 9; and consequently, whenever m exceeds 9, the points will cease to be entirely arbitrary. Thus, in the case of a cubic surface having its full number of arbitrary constants, the foregoing formula would give $m = 10$; and there would be a condition to be fulfilled by the coordinates of one of the points, viz., that it should lie on the quadric drawn through the other nine. But there are many particular surfaces of degrees higher than 2, whose equations do not contain the full number of constants; and the statement above made will have application to them without any limitation in the arbitrary position of the points, provided that their number does not exceed 9.

As an example, we may take the case of a cubic surface having no multiple point; the equation to this surface may be put into the form

$$ax^3+by^3+cz^3+dt^3+eu^3=0,$$

where $$x+y+z+t+u=0.$$

The four planes x, y, z, t may be chosen arbitrarily, and of the four remaining constants $a:b:c:d:e$, three may be determined by means of the equations

$$ax^3+\ldots=0, \quad ax_1^3+\ldots=0, \quad ax_2^3+\ldots=0;$$

and the equation $(0, 1, 2)=0$ will then take the form $(d, e)^3=0$. Hence through any three points in space three cubic surfaces, having no multiple point, and having their four common fundamental planes chosen arbitrarily, may be drawn such that they may be touched by a quadric at the three points.

More generally,

Through any three (or four) points of space three surfaces, having four (or five) independent constants in their equation, can be drawn such that a quadric may be described touching any one of the surfaces in the three (or in three out of the four) points.

Through any four (or five) points of space nine surfaces, having six (or seven) independent constants in their equation, can be drawn such that a quadric may be described touching any one of the surfaces in the four (or in four out of the five) points.

Or still more generally, if P, P_1, ... be any m (or $m+1$) points lying on any quadric whatsoever, then

Through any m (or $m+1$) such points of space 3^{m-2} surfaces, having $2m-2$ (or $2m-1$) independent constants in their equation, can be drawn such that a quadric may be described touching any one of the surfaces in the m (or in m out of the $m+1$) points; m being, as explained above, $\geqslant 9$.

Thus, for example: The equation of a quartic scroll having a triple line is $$(ax+by)\,zx^2+(cx+dy)\,wy^2-mx^2y^2=0.$$

Hence, Through any three points of space three quartic scrolls, having the same double line, can be drawn such that a quadric may be described touching any one of the scrolls in the three points.

Again, The equation of a quartic surface having for its nodal line the twisted cubic

$$p=xz-y^2=0, \quad q=xw-yz=0, \quad r=yw-z^2=0,$$

may be put in the form

$$ap^2+bq^2+cr^2+2\,(fqr+grp+hpq)=0.$$

Hence, Through any four points of space three quartics, having the same twisted cubic for their common nodal line, may be drawn such that a quadric may be described touching any one of the quartics in three of the points.

As a last example, The equation of a quartic scroll, having two non-intersecting double lines, is of the form

$$(a, b, c, f, g, h, l, m, n)(x^2, 2xy, y^2)(z^2, 2zw, w^2) = 0.$$

Hence, Through any five points of space 27 quartic scrolls, having the same non-intersecting pair of double lines, may be drawn such that a quadric may be described touching any one of the scrolls in the five points.

Note on the Inversion of Bernoulli's Theorem in Probabilities.

By C. J. Monro, M.A.

[*Read February 12th*, 1874.]

1. Under the name of Bernoulli's Theorem are comprehended two theorems which, with a little licence, we may distinguish as the deductive and the inductive. The deductive theorem assumes the constant probability p of a given result on a single trial, and determines the probability P that on m trials the result will be produced from $mp-l$ to $mp+l$ times, or from $x-l$ to $x+l$ if x is the greatest integer in $mp+p$. The inductive theorem assumes that the given result is produced mp times on m trials which give a constant facility for its production (that is, are made under definable circumstances which, if defined, would give a constant probability for the same), and determines P′ the probability that this facility lies between $p \pm \frac{l}{m}$. In both theorems l is of the order $\sqrt{m}$ at highest, and it is supposed that terms of the order $\frac{1}{m}$ may be rejected.

2. There is no question about the value of P. If h stand for $\frac{1}{2p(1-p)m}$, the value found by Laplace may be expressed in the form

$$2\sqrt{\frac{h}{\pi}} \cdot \int_0^{l+\frac{1}{2}} e^{-h\lambda^2}\, d\lambda.$$

For P′ Laplace found two values. In No. 16, Book ii., he makes $P' = P$ by an implicit inference from the deductive theorem, or, as Mr. Todhunter says, by an "assumed inversion" of it. But in No. 26, by a direct calculation which presupposes that a variable of which nothing is known has equally probable values within equal ranges, he finds the proper value $2\sqrt{\frac{h}{\pi}} \int_0^{l} e^{-h\lambda^2}\, d\lambda.$

3. The discrepancy, which, when the value of P is expressed as

above,* affects the limits of integration only, has been the subject of some discussion in this country. (See Mr. Todhunter's "History," § 997.) Adopting the result of Laplace's direct investigation, I hope to show that, upon the hypothesis of equally probable values within equal ranges, the inversion is so far legitimate, that either theorem may be inferred from the other with little calculation, and in particular without the approximate evaluation of a general integral, and accordingly that the two solutions are identical in principle.

4. It is obvious that $l+\frac{1}{2}$ might be substituted for l in the statement of the deductive theorem in § 1; because, though the approximative process introduces the integration of a differential, the problem is to integrate a finite difference depending on integer values of λ, so that *from* $-l$ *to* $+l$ is the same thing as *from* $-(l+\theta)$ *to* $+(l+\theta')$, if θ and θ' are any positive proper fractions. Therefore it is not without special justification to be presumed, when, in correlating the inductive with the deductive theorem, we seek the corresponding limits of its continuously varying argument, that θ and θ' are indefinitely small.

Integration of a differential is essential to the inductive theorem. Here, as usual, the differential is given as the product of a finite factor and the differential of the independent variable. Correspondence ought to exist with respect to both factors; and this is secured by making the finite difference (unity) of the independent variable a factor of the quantity integrated in the finite summation for the deductive theorem. There must therefore be one such factor to every term, so that $2l+1$, and not $2l$, is the range of the independent variable. Therefore $\theta+\theta'=1$, and for the required degree of approximation it is useless to discuss values differing from $\theta=\theta'=\frac{1}{2}$.† Therefore, in the inductive

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* Besides being more convenient for the present purpose, $\int_0^{x+\frac{1}{2}} u_x\,dx$ is a little more exact than $\int_0^x u_x\,dx + \frac{u_x}{2}$ as an approximation to Σu_{x+1}. For while

$$\Sigma u_{x+1} = \int_0^x u_x\,dx + \frac{u_x}{2} + \frac{1}{12}\cdot\frac{du_x}{dx} + \ldots, \quad \text{by Euler's theorem,}$$

we have also

$$\int_0^{x+\frac{1}{2}} u_x\,dx = \int_0^x u_x\,dx + \frac{u_x}{2} + \frac{1}{8}\cdot\frac{du_x}{dx} + \frac{1}{48}\cdot\frac{d^2u_x}{dx^2} + \ldots, \quad \text{by Taylor's;}$$

therefore

$$\Sigma u_{x+1} = \int_0^{x+\frac{1}{2}} u_x\,dx - \frac{1}{24}\cdot\frac{du_x}{dx} - \ldots.$$

† If we consider P as $2\psi(o)$ substituted for a true value $\psi\left(\frac{\theta-\theta'}{2}\right)+\psi\left(\frac{-\theta+\theta'}{2}\right)$, the error committed is $-\frac{(\theta-\theta')^2}{4}\psi''(o)$, or $\frac{(\theta-\theta')^2}{4\sqrt{\pi}}\cdot h\sqrt{h}\,(2l+1)\,e^{-h(l+\frac{1}{2})^2}$, which is of the order $\frac{1}{m}$.

theorem, P is the probability that the facility lies between the limits $p \pm \frac{l+\frac{1}{2}}{m}$, and the second solution is correct for the limits $p \pm \frac{l}{m}$; provided always that a valid correspondence exists between the two theorems.

5. The existence of such correspondence, and therefore, I think, the legitimacy of the inversion, is easily shown. Let u_n be the probability of n $(=m\varpi)$ results from m trials each of a facility $\frac{x}{m}$, and suppose $\mathrm{U}_n d\varpi$ to be the probability given by x results upon m trials that their facility, assumed constant, is within $+d\varpi$ of ϖ $\left(=\frac{n}{m}\right)$. This supposition expresses the hypothesis of equally probable values of the facility within equal ranges. Then we have to compare u_n in the deductive theorem with $\int \mathrm{U}_n d\varpi$ taken between $\frac{n \pm \frac{1}{2}}{m}$ in the inductive.

If $n+n' = x+x' = m$, we have

$$u_n = \frac{\lfloor m}{\lfloor n \, \lfloor n'} \frac{x^n x'^{n'}}{m^m},$$

$$\mathrm{U}_n = \frac{\varpi^x (1-\varpi)^{x'}}{\int_0^1 \varpi^x (1-\varpi)^{x'} d\varpi} = \frac{\lfloor m+1}{\lfloor x \, \lfloor x'} \frac{n^x n'^{x'}}{m^m};$$

so that

$$\frac{\mathrm{U}_n}{(m+1)\, u_n} = \frac{\lfloor n}{\lfloor x} \frac{n^x}{x^n} \frac{\lfloor n'}{\lfloor x'} \frac{n'^{x'}}{x'^{n'}}.$$

And if $n = x+\lambda$, and therefore $n' = x'-\lambda$, the transformation employed by Laplace for the deductive theorem yields

$$\log \lfloor n \, n^x - \log \lfloor x \, x^n = \lambda + \left(\frac{\lambda}{2x} + \frac{\lambda^3}{6x^2}\right) + \dots,$$

$$\log \lfloor n' n'^{x'} - \log \lfloor x' x'^{n'} = -\lambda - \left(\frac{\lambda}{2x'} + \frac{\lambda^3}{6x'^2}\right) - \dots;$$

therefore $$\log \frac{\mathrm{U}_n}{(m+1)\, u_n} = \frac{\lambda}{2}\left(\frac{1}{x} - \frac{1}{x'}\right) + \frac{\lambda^3}{6}\left(\frac{1}{x^2} - \frac{1}{x'^2}\right) + \dots,$$

which is of the order $\frac{1}{\sqrt{m}}$, but does not appear in the double term formed by taking into account equal and opposite values of λ. That is, when terms of the order $\frac{1}{m}$ are rejected, $\mathrm{U}_n = (m+1)\, u_n$, and $\int \mathrm{U}_n d\varpi = (m+1) \int u_n d\varpi$; which, taken between $\frac{n \pm \frac{1}{2}}{m}$, differs from $\frac{m+1}{m} u_n$, or $\left(\frac{1}{m}\right.$ being neglected$\left.\right)$ from u_n, by a relative error less

than that incurred by taking u_n for the arithmetic mean of the values $u_{n\pm1}$ which enclose a double range. But $\frac{u_{n\pm1}}{u_n}$ are equal respectively to $\frac{n'}{n+1}\cdot\frac{x}{x'}$ and $\frac{n}{n'+1}\cdot\frac{x'}{x}$, or $\left(\frac{1}{n}, \frac{1}{n'}\text{ being neglected}\right)$ to $\frac{n'x}{nx'}$ and $\frac{nx'}{n'x}$. Therefore

$$\frac{\frac{1}{2}(u_{n+1}+u_{n-1})-u_n}{u_n} = \frac{(n'x-nx')^2}{2nn'xx'} = \frac{m^2\lambda^2}{2nn'xx'},$$

which is of the order $\frac{1}{m}$ at highest; and this error, being relative to u_n, does not rise in order by accumulation on summing.

Therefore the values for the two theorems are equal double term by double term. And we find this, first, it is true, by taking an integral of the same differential as in the usual investigations of the inductive theorem, but only its definite integral between 0 and 1; and, secondly, by the transformation which Laplace had employed for the deductive theorem just before introducing the inversion. Perhaps it may be inferred (however we account for the form in which he has stated the result of this process) that Laplace made no mere assumption in performing it, but intended it as a corollary from what had gone before in virtue of reasoning which might be expressed in the argument above. But more explicit writers who adopt this corollary as a complete proof ought to say at what point they introduce the hypothesis of equal probability within equal range.

6. Before quitting this subject, it is worth while to take notice of a point of elementary algebra connected with it, which has been somewhat imperfectly stated by eminent writers.

It is sometimes said* that, x and x' being whole numbers, the quantity $\frac{\lfloor x+x'}{\lfloor x \lfloor x'} p^x p'^{x'}$ is greatest when x, x' are as nearly as possible in the ratio of p, p'; or if $p+p'=1$, and $x+x'=m$, when x, x' are as nearly as possible equal to mp, mp'. These statements, it should be observed, are made in dealing with large numbers, to which they are properly applicable; but to include other cases some modification is requisite. The fact is that $mp-x$, which, lying between $-p$ and $+p'$, has a range of unity,

* De Morgan, Theory of Probabilities, in the Encycl. Metr. (1837), § 27.
Galloway, Probability, in the Encycl. Brit., § 88.
Fries, Versuch einer Kritik der Principien der Wahrscheinlichkeitsrechnung (1842), p. 52.
Cournot, Exposition de la Théorie des Chances et des Probabilités (1843), p. 55.
Bunyakovski, Osnovaniya &c. (Principles of the Mathematical Theory of Probabilities, Petersburg, 1846), p. 31.
Todhunter, History of the Theory of Probability (1865), p. 548.

may yet be outside the limits $\pm\frac{1}{2}$. When this is so, then (as the case may be) $x\pm1$ is nearer to mp than x is. Thus, in the expansion of $(4+1)^8$ the first three terms are as 4, 8, 7, so that the second is the greatest; but the third has the exponents (6 and 2) most nearly equal to $6+\frac{2}{5}$ and $2-\frac{2}{5}$, which are 8 times $\frac{4}{5}$ and $\frac{1}{5}$ respectively.

However, as $(m+1)p-x$ is positive and less than unity, $(m+1)p-(x+\frac{1}{2})$ is within the limits $\pm\frac{1}{2}$, so that $x+\frac{1}{2}$ and $x'+\frac{1}{2}$ are as nearly as possible equal to $(m+1)p$ and $(m+1)p'$; therefore it is true that, *when increased by* $\frac{1}{2}$, the exponents of the greatest term of the expansion are as nearly as possible in the ratio of the corresponding terms of the binomial.

On the Transformation of Continued Products into Continued Fractions. *By* J. W. L. Glaisher, M.A.

[*Read January 8th*, 1874.]

1. The present paper had its origin in a very remarkable continued fraction for π given by Professor Sylvester in the "Philosophical Magazine" for 1869,* viz.,

$$\frac{\pi}{2}=1+\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+}\,\frac{4.5}{1+\&c.} \quad \text{.....(i.),}$$

which is there shown to be equivalent to Wallis's formula,

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}.$$

It was not, however, by a direct transformation of the product that Professor Sylvester obtained his result, but by means of a complete solution that he had previously found of the equation of finite differences $u_{x+1}=\frac{u_x}{x}+u_{x-1}$; so that a direct proof of (i.) by ordinary algebra seemed likely to be interesting. Further, whenever we obtain an expression for π as a series, product, or continued fraction, it is pretty safe to assume that the formula admits of generalisation; viz., that there exists a corresponding series, &c., involving $\sin^{-1}x$ or some other expression which, for a particular value of the variable, becomes equal to π. In the present instance, as the continued fraction is a transformation of Wallis's formula, it seemed desirable to enquire what was the fraction corresponding to the expression for $\sin x$ as an infinite

* "Note on a new Continued Fraction applicable to the Quadrature of the Circle," Phil. Mag., S. 4, vol. xxxvii., pp. 373—375 (May, 1869).

product of which Wallis's formula is a particular case. This naturally led to the investigation of a direct rule to transform any product into a continued fraction, and it is convenient to commence with the general formulæ.

2. It is easily seen that

$$\frac{\beta}{\alpha}+\frac{\beta\beta'}{\alpha\alpha'}+\frac{\beta\beta'\beta''}{\alpha\alpha'\alpha''}+\&c.=\frac{\beta}{\alpha-}\ \frac{\alpha\beta'}{\alpha'+\beta'-}\ \frac{\alpha'\beta''}{\alpha''+\beta''-\&c.}\ \ldots\ldots\text{(ii.)}.$$

This may be regarded as a well-known theorem, as it is used, in one shape or another, in several memoirs in "Crelle"; and the case, when the β's are all equal to unity, and the terms of alternate sign, is proved in Bertrand's "Calcul Différentiel." The demonstration is very simple: $1+\frac{\beta'}{\alpha'}=\frac{1}{1-\frac{\beta'}{\alpha'+\beta'}}$; and to include the next term we have merely to replace $\frac{1}{\alpha'}$ by $\frac{1}{\alpha'}\left(1+\frac{\beta''}{\alpha''}\right)$, viz., to replace α' by $\alpha'-\frac{\alpha'\beta''}{\alpha''+\beta''}$. It is thus evident that the first n terms of the series are identically equal to the continued fraction taken as far as to the end of the nth quotient.

By a change of sign we have

$$\frac{\beta}{\alpha}-\frac{\beta\beta'}{\alpha\alpha'}+\frac{\beta\beta'\beta''}{\alpha\alpha'\alpha''}-\&c.=\frac{\beta}{\alpha+}\ \frac{\alpha\beta'}{\alpha'-\beta'+}\ \frac{\alpha'\beta''}{\alpha''-\beta''+\&c.}\ \ldots\ldots\text{(iii.)}.$$

The following are obvious algebraical identities:

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots=1+\frac{b}{a}+\frac{(a+b)b'}{aa'}+\frac{(a+b)(a'+b')b''}{aa'a''}+\ldots\ \ldots\text{(iv.)}$$

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots=1+\frac{b}{a}-\frac{(a+b)b'}{aa'}+\frac{(a+b)(a'-b')b'}{aa'a''}-\ldots\ \ldots\text{(v.)}$$

$$\left(1-\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1-\frac{b'}{a''}\right)\ldots=1-\frac{b}{a}+\frac{(a-b)b'}{aa'}-\frac{(a-b)(a'+b')b''}{aa'a''}+\ldots\ \ldots\text{(vi.)}$$

$$\left(1-\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots=1-\frac{b}{a}-\frac{(a-b)b'}{aa'}-\frac{(a-b)(a'-b')b''}{aa'a''}-\ldots\ \ldots\text{(vii.)}$$

By means of formulæ (ii.)—(vii) any continued product can be converted into a continued fraction such that n factors of the former shall be equivalent to the nth convergent to the latter [or to the $(n+1)$th convergent, or to the $(n-1)$th, or the $(n-2)$th, &c., as we please].

Thus, from (ii.) and (iv.),

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\left(1+\frac{b'''}{a'''}\right)\ldots$$
$$=1+\frac{\beta}{\alpha-}\ \frac{\alpha\beta'}{\alpha'+\beta'-}\ \frac{\alpha'\beta''}{\alpha''+\beta''-}\ \frac{\alpha''\beta'''}{\alpha'''+\beta'''-\&c.}\ \ldots\text{(viii.)},$$

where
$$\begin{aligned} \beta &= b, & \alpha &= a,\\ \beta' &= (a+b)\,b', & \alpha' &= a'b,\\ \beta'' &= (a'+b')\,b'', & \alpha'' &= a''b',\\ \beta''' &= (a''+b'')\,b''', & \alpha''' &= a'''b'',\\ &\text{\&c.}, & &\text{\&c.};\end{aligned}$$

and from (iii.) and (v.),

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots = 1+\frac{\beta}{\alpha+}\ \frac{\alpha\beta'}{a'-\beta'+}\ \frac{a'\beta''}{a''-\beta''+\text{\&c.}}\ldots\text{(ix.)},$$

where
$$\begin{aligned} \beta &= b, & \alpha &= a,\\ \beta' &= (a+b)\,b', & \alpha' &= a'b,\\ \beta'' &= (a'-b')\,b'', & \alpha'' &= a''b',\\ &\text{\&c.}, & &\text{\&c.}\end{aligned}$$

The last two formulæ [viz., (viii.) and (ix.)] are those which will be chiefly used. The other two that follow from (vi.) and (vii.) are

$$\left(1-\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots = 1-\frac{\beta}{\alpha+}\ \frac{\alpha\beta'}{a'-\beta'+\text{\&c.}}\ \ldots\ldots\text{(x.)},$$

where
$$\begin{aligned} \beta &= b, & \alpha &= a,\\ \beta' &= (a-b)\,b', & \alpha' &= a'b,\\ \beta'' &= (a'+b')\,b'', & \alpha'' &= a''b',\\ &\text{\&c.}, & &\text{\&c.};\end{aligned}$$

and
$$\left(1-\frac{b}{a}\right)\left(1-\frac{b'}{a'}\right)\left(1-\frac{b''}{a''}\right)\ldots = 1-\frac{\beta}{\alpha-}\ \frac{\alpha\beta'}{a'+\beta'-\text{\&c.}}\ \ldots\ldots\text{(xi.)},$$

where
$$\begin{aligned} \beta &= b, & \alpha &= a,\\ \beta' &= (a-b)\,b', & \alpha' &= a'b,\\ \beta'' &= (a'-b')\,b'', & \alpha'' &= a''b',\\ &\text{\&c.}, & &\text{\&c.}\end{aligned}$$

The special case of $b=b'=b''=\text{\&c.}=x$ deserves notice: thus,

$$\left(1+\frac{x}{a}\right)\left(1+\frac{x}{a'}\right)\left(1+\frac{x}{a''}\right)\ldots = 1+\frac{x}{a}+\frac{x\,(x+a)}{aa'}+\frac{x\,(x+a)\,(x+a')}{aa'a''}+\ldots$$
$$= 1+\frac{x}{a-}\ \frac{a\,(x+a)}{x+a+a'-}\ \frac{a'(x+a')}{x+a'+a''-\text{\&c.}}\ \ldots\ldots\text{(xii.)};$$

and in particular, when $x=1$,

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a'}\right)\left(1+\frac{1}{a''}\right)\ldots = 1+\frac{1}{a-}\ \frac{a\,(a+1)}{a'+a+1-}\ \frac{a'(a'+1)}{a''+a'+1-\text{\&c.}}.$$

4. In all the formulæ (viii.)—(xii.), the first n factors of the product are identically equal to the right-hand side when the continued fraction is replaced by the nth convergent to it. In all cases it is easy to exhibit the right-hand side wholly as a continued fraction, for

$$1+\frac{\beta}{\alpha\pm Q}=\frac{1}{1-\dfrac{\beta}{\alpha+\beta\pm Q}},$$

and

$$1-\frac{\beta}{\alpha\pm Q}=\frac{1}{1+\dfrac{\beta}{\alpha-\beta\pm Q}}.$$

Thus, for example, from (viii.)

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b'}{a'}\right)\left(1+\frac{b''}{a''}\right)\ldots=\frac{1}{1-}\ \frac{\beta}{\alpha+\beta-}\ \frac{\alpha\beta'}{\alpha'+\beta'-}\ \frac{\alpha'\beta''}{\alpha''+\beta''-\&c.},$$

This form is rather more elegant in the case of (viii.) and (x.), but it violates the regularity of the signs in (ix.) and (xi.); thus (ix.) gives

$$\left(1+\frac{b}{a}\right)\left(1-\frac{b}{a}\right)\left(1+\frac{b''}{a''}\right)\ldots=\frac{1}{1-}\ \frac{\beta}{\alpha+\beta+}\ \frac{\alpha\beta}{\alpha'-\beta'+\&c.};$$

the form adopted seems therefore preferable, viz., that in which the results are of the form $1\pm$continued fraction.

5. I now proceed to apply the formulæ obtained to the case of the infinite products for the sine and cosine; we have

$$\frac{\sin\pi x}{\pi x}=(1+x)(1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\left(1-\frac{x}{3}\right)\ldots,$$

whence

$$\frac{\pi}{n\sin\frac{\pi}{n}}=\left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{2n-1}\right)\left(1-\frac{1}{2n+1}\right)\ldots$$

$$=1+\frac{1}{n-1+}\ \frac{(n-1)\,n}{1+}\ \frac{n\,(n+1)}{n-1+}\ \frac{(2n-1)\,2n}{1+}\ \frac{2n\,(2n+1)}{n-1+\&c.}\ldots\ \text{(xiii.)},$$

the denominators being alternately unity and $n-1$. Sylvester's fraction (i.), viz. $\frac{\pi}{2}=1+\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{1+}\ \frac{3.4}{1+\&c.}$ is the particular case when $n=2$, and (xiii.) is the general formula for the sine referred to in § 1. We see at once that Sylvester's fraction is the only one deducible from (xiii.), in which the denominators are all identical; but by giving different values to n we obtain a set of continued fractions with alternating denominators; thus, putting $n=3$, 4, and 6, we have

$$\frac{2\pi}{3\sqrt{3}}=1+\frac{1}{2+}\ \frac{2.3}{1+}\ \frac{3.4}{2+}\ \frac{5.6}{1+}\ \frac{6.7}{2+}\ \frac{8.9}{1+}\ \frac{9.10}{2+\&c.},$$

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3+}\ \frac{3.4}{1+}\ \frac{4.5}{3+}\ \frac{7.8}{1+}\ \frac{8.9}{3+}\ \frac{11.12}{1+}\ \frac{12.13}{3+\&c.},$$

$$\frac{\pi}{3}=1+\frac{1}{5+}\ \frac{5.6}{1+}\ \frac{6.7}{5+}\ \frac{11.12}{1+}\ \frac{12.13}{5+}\ \frac{17.18}{1+}\ \frac{18.19}{5+\&c.}.$$

From the cosine series we have

$$\sec\frac{\pi}{2n} = \left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{3n-1}\right)\left(1-\frac{1}{3n+1}\right)\cdots$$

$$= 1+\frac{1}{n-1+}\ \frac{(n-1)\,n}{1+}\ \frac{n\,(n+1)}{2n-1+}\ \frac{(3n-1)\,3n}{1+}\ \frac{3n\,(3n+1)}{2n-1+\&c.} \quad \text{...(xiv.)}$$

Take $n=2$ and 3, and there result

$$\sqrt{2} = 1+\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{3+}\ \frac{5.6}{1+}\ \frac{6.7}{3+}\ \frac{9.10}{1+}\ \frac{10.11}{3+\&c.},$$

$$\frac{2}{\sqrt{3}} = 1+\frac{1}{2+}\ \frac{2.3}{1+}\ \frac{3.4}{5+}\ \frac{8.9}{1+}\ \frac{9.10}{5+}\ \frac{14.15}{1+}\ \frac{15.16}{5+\&c.}.$$

If we put $n=1$ in (xiv.) as it stands, we merely obtain $\infty=\infty$, but if we take $n=1+\epsilon$ (ϵ being infinitesimal), replace $\cos\frac{1}{2}\pi\,(1-\epsilon)$ by $\frac{1}{2}\pi\epsilon$, and divide out by ϵ, we deduce Sylvester's formula over again. We may also obtain results by giving n fractional values, *e.g.*, take $n=\frac{3}{2}$.

6. For comparison with the above continued fractions for the sine and cosine, it is interesting to express in the same way several other series and products which also are equivalent to these functions; thus

$$\frac{\sin\pi x}{\pi x} = (1+x)(1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\left(1-\frac{x}{3}\right)\cdots$$

$$= 1+\frac{x}{1-}\ \frac{1+x}{x-}\ \frac{1-x}{1+x-}\ \frac{2\,(2+x)}{x-}\ \frac{2\,(2-x)}{1+x-\&c.} \quad \text{.....(xv.),}$$

whence $$\frac{2}{\pi} = 1+\frac{1}{2-}\ \frac{2.3}{1-}\ \frac{2.1}{3-}\ \frac{4.5}{1-}\ \frac{4.3}{3-}\ \frac{6.7}{1-}\ \frac{6.5}{3-\&c.}.$$

Similarly $$\cos\frac{\pi x}{2} = 1+\frac{x}{1-}\ \frac{1+x}{x-}\ \frac{1-x}{2+x-}\ \frac{3\,(3+x)}{x-\&c.} \quad \text{.....(xvi.)}$$

Again, we have

$$\frac{\sin\pi x}{\pi x} = (1-x^2)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots$$

$$= 1-\frac{x^2}{1^2}-\frac{x^2(1-x^2)}{1^2.2^2}-\frac{x^2(1-x^2)(2^2-x^2)}{1^2.2^2.3^2}-\cdots$$

$$= 1-\frac{x^2}{1-}\ \frac{1-x^2}{1+2.2-x^2-}\ \frac{2^2(2^2-x^2)}{1+3.4-x^2-}\ \frac{3^2(3^2-x^2)}{1+4.6-x^2-\&c.}\cdots\text{(xvii.),}$$

the denominators being of the form $1+n\,.\,2(n-1)-x^2$.

If herein we take $x=1+\epsilon$, ϵ being infinitesimal, expand the left-hand side, and divide out by ϵ, we obtain, after some reductions,

$$2 = \frac{2^2.1.3}{3.4-}\ \frac{3^2.2.4}{6.4-}\ \frac{4^2.3.5}{8.5-\&c.},$$

which at once reduces to $1=\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-\&c.}$, and serves as a verification.

Similarly, from the cosine series we have

$$\cos\tfrac{1}{2}\pi x=1-\frac{x^2}{1-}\,\frac{1-x^2}{8.1^2+2-x^2-}\,\frac{3^2(3^2-x^2)}{8.2^2+2-x^2-}\,\frac{5^2(5^2-x^2)}{8.3^2+2-x^2-\&c.},$$

the denominators being of the form $8n^2+2-x^2$.

Also, from $\sin\tfrac{1}{2}\pi x=x-\frac{x(x^2-1^2)}{1.2.3}+\frac{x(x^2-1^2)(x^2-3^2)}{1.2.3.4.5}-\ldots$

we deduce

$$\frac{\sin\frac{1}{2}\pi x}{x}=1+\frac{1-x^2}{2.3-}\,\frac{2.3\,(3^2-x^2)}{3^2+4.5-x^2-}\,\frac{4.5\,(5^2-x^2)}{5^2+6.7-x^2-\&c.}.$$

Taking herein $x=1+\epsilon$ as before, there results

$$1=\frac{1}{3-}\,\frac{2.3}{7-}\,\frac{4.5}{11-}\,\frac{6.7}{15-}\,\frac{8.9}{19-\&c.}\qquad\ldots\ldots\ldots\ldots\text{(xviii.)}$$

From $\cos\tfrac{1}{2}\pi x=1-\frac{x^2}{1.2}+\frac{x^2(x^2-2^2)}{1.2.3.4}-\frac{x^2(x^2-2^2)(x^2-4^2)}{1.2.3.4.5.6}+\ldots$

we deduce

$$\cos\tfrac{1}{2}\pi x=1-\frac{x^2}{1.2-}\,\frac{1.2\,(2^2-x^2)}{2^2+3.4-x^2-}\,\frac{3.4\,(4^2-x^2)}{4^2+5.6-x^2-\&c.};$$

where, putting $x=1$,

$$1=\frac{1.2}{5-}\,\frac{3.4}{9-}\,\frac{5.6}{13-}\,\frac{7.8}{17-\&c.}.$$

7. The formulæ

$$\cos\tfrac{1}{3}\pi x=1-\frac{x^2}{1.2}+\frac{x^2(x^2-1^2)}{1.2.3.4}-\frac{x^2(x^2-1^2)(x^2-2^2)}{1.2.3.4.5.6}+\ldots$$

$$\sin\tfrac{1}{3}\pi x=\tfrac{1}{2}\sqrt{3}\left\{x-\frac{x(x^2-1^2)}{1.2.3}+\frac{x(x^2-1^2)(x^2-2^2)}{1.2.3.4.5}-\ldots\right\}$$

yield similar continued fractions to those written down in § 6.

If in the first we put $x=1+\epsilon$ as before, we find

$$\frac{\pi}{\sqrt{3}}=2-\frac{1}{6-}\,\frac{1.6}{11-}\,\frac{2.10}{16-}\,\frac{3.14}{21-}\,\frac{4.18}{26-\&c.}.$$

The ordinary series for $\sin x$ and $\cos x$, viz. $x-\frac{x^3}{1.2.3}+\ldots$ and $1-\frac{x^2}{1.2}+\ldots$, when transformed into continued fractions, give

$$\frac{\sin x}{x}=1-\frac{x^2}{2.3+}\,\frac{2.3x^2}{4.5-x^2+}\,\frac{4.5x^2}{6.7-x^2+\&c.},$$

and

$$\cos x=1-\frac{x^2}{1.2+}\,\frac{1.2x^2}{3.4-x^2+}\,\frac{3.4x^2}{5.6-x^2+\&c.},$$

and possess no special point of interest.

A number of series to which the methods used in this paper may be applied will be found in two papers, "On the Deduction of Series from Infinite Products," and "Remarks on certain Series occurring in a paper 'On the Deduction of Series from Infinite Products,'" "Messenger of Mathematics," New Series, vol. ii., pp. 138–142 and 153–157; but it is not worth while to give these applications here, as they present no novelty, and are generally less interesting than those already noticed.

8. It is perhaps worth remarking that every equation connecting a product and a continued fraction gives a numerical identity; *e.g.*, from (xvii.), if n be any integer,

$$1-\frac{n^2}{1-}\ \frac{1-n^2}{1+2.2-n^2-}\ \frac{2^2(2^2-n^2)}{1+3.4-n^2-}\ \cdots\cdots\ \frac{(n-1)^2\{(n-1)^2-n^2\}}{1+n.2(n-1)-n^2}=0.$$

If the factors of a product be taken in a different order, the equivalent continued fraction is altered in a corresponding manner; but the results, of course, take the most elegant form when the factors are so arranged that the denominators are in order of magnitude. If in (xiii.), (xiv.), (xv.), and (xvi.) the order of every pair of factors be inverted, the effect is merely to change the sign of n and x.

From (xii.) we have the identity

$$\left(1-\frac{x}{a-}\ \frac{a(a-x)}{a+a'-x-}\ \frac{a'(a'-x)}{a'+a''-x-\&c.}\right)\left(1+\frac{x}{a-}\ \frac{a(a+x)}{a+a'+x-}\cdot\frac{a'(a'+x)}{a'+a''+x-\&c.}\right)$$
$$=1-\frac{x^2}{a^2-}\ \frac{a^2(a^2-x^2)}{a^2+a'^2-x^2-}\ \frac{a'^2(a'^2-x^2)}{a'^2+a''^2-x^2-\&c.},$$

which is perhaps worth notice.

One chief use of expressing series &c. as continued fractions is that sometimes we are enabled thereby to infer their irrationality by Lambert's principle; but it will be seen that Lambert's method generally gives no result when applied to the cases discussed in this paper.

It is not unlikely that by simple transformations, such as

$$a-\frac{1}{b}=a-1+\frac{1}{1+\frac{1}{b-1}},$$

most of the numerical results [*e.g.*, (xviii.)] may be reduced to forms in which their truth is evident; but I have not investigated the question.

9. There is a remarkable fact connected with the continued fraction

$$\frac{\pi}{2}=1+\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{1+}\ \frac{3.4}{1+\&c.},$$

to which Professor Sylvester draws attention in the following words:—
"At first sight it might seem as if the above-stated continued fraction

were incapable of teaching anything that cannot be got direct out of the Wallisian representation itself, that has become transformed into it. . . . . But I think a substantial difference does arise in favour of the continued fraction form, inasmuch as it indicates a certain obvious correction to be applied in order that the convergence may become more exact. For if we call

$$\frac{n(n+1)}{1+}\ \frac{(n+1)(n+2)}{1+}\ \ldots \text{ ad infinitum } u_n,$$

we have $u_n = \dfrac{n^2+n}{1+u_{n+1}}$. This shows than u_n cannot remain finite when n becomes infinite; for then u_{n+1} would also be finite, and consequently u_n would be a finite fraction of infinity, which is a contradiction in terms.

"Hence ultimately $u_n u_{n+1} = n^2+n$, *i. e.*, $u_n = n$. . . . . Thus we may write, when n is very large,

$$\frac{\pi}{2} = 1+\frac{1}{1+}\ \frac{2}{1+}\ \frac{6}{1+}\ \ldots \frac{n^2-n}{1+n}\,."$$

Taking $n=4$ and 5, Professor Sylvester shows the advantage of taking account of the value of u_n.

To examine the nature of this correction, we recur to (v.) and (iii.), and find

$$u = \frac{2.2.4.4.6.6\ \ldots\ 2m\ .\ 2m}{1.3.3.5.5.7\ \ldots\ (2m-1)(2m+1)}$$

$$= (1+1)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots\cdots$$

$$\cdots\cdots\left(1+\frac{1}{2m-1}\right)\left(1-\frac{1}{2m+1}\right)$$

$$= 1+1-\frac{2}{3}+\frac{2^2}{3^2}-\frac{2^2.4}{3^2.5}+\frac{2^2.4^2}{3^2.5^2}-\cdots\cdots$$

$$\cdots\cdots+\frac{2^2.4^2\ldots(2m-2)^2}{3^2.5^2\ldots(2m-1)^2}-\frac{2^2\ldots(2m-2)^2\,2m}{3^2.5^2\ldots(2m-1)^2\,(2m+1)}$$

$$= 1+\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{1+\ldots}\ \frac{(n-1)n}{1},$$

if we take $n=2m$. (We might, of course, just as well have included $2m+1$ terms in u, and taken $n=2m+1$.)

Professor Sylvester's correction consists in taking the last quotient in the continued fraction just written to be $\dfrac{(n-1)n}{1+n}$ instead of $\dfrac{(n-1)n}{1}$; viz., it consists in taking the last factor in the denominator of the last term of the series to be $4m+1$ instead of $2m+1$, and therefore in taking the last factor in the product u to be $\dfrac{4m}{4m+1}$ instead of $\dfrac{2m}{2m+1}$.

In order to include the remaining terms of the series, we must replace the factor $\frac{1}{2m+1}$ in the last term by

$$\frac{1}{2m+1}\left\{1-\frac{2m}{2m+1}+\frac{2m\,(2m+2)}{(2m+1)(2m+3)}-\cdots\right\},$$

and we see that we make a step towards including these terms by replacing the quantity within brackets by the geometrical progression

$$1-\frac{2m}{2m+1}+\left(\frac{2m}{2m+1}\right)^2-\cdots,$$

viz., by replacing $\frac{1}{2m+1}$ by $\frac{1}{2m+1}\cdot\frac{2m+1}{4m+1}$, that is, by $\frac{1}{4m+1}$.

If we call the quantity within the brackets v_{2m}, we have

$$v_{2m}=1-\frac{2m}{2m+1}\,v_{2m+1};$$

and if we assume that, when m is large, we may take $v_{2m}=v_{2m+1}$, we get $v_{2m}=\frac{2m+1}{4m+1}$, as before, and thence the same result. We therefore see that we do obtain the required correction from the series, though in not so satisfactory a manner as from the continued fraction.

The correction, when applied to the product, amounts to replacing $\frac{2m}{2m+1}\cdot\frac{2m+2}{2m+1}\cdot\frac{2m+2}{2m+3}\cdot\frac{2m+4}{2m+3}\cdots$ by $\frac{4m}{4m+1}$. That this quantity is a first approximation to the value of the product just written, may be shown as follows. Let q denote the product; then

$$\begin{aligned}\log q&=\log\left(1-\frac{1}{2m+1}\right)+\log\left(1+\frac{1}{2m+1}\right)+\log\left(1-\frac{1}{2m+3}\right)+\cdots\\&=\log\left\{1-\frac{1}{(2m+1)^2}\right\}+\log\left\{1-\frac{1}{(2m+3)^2}\right\}+\cdots\\&=-\frac{1}{(2m+1)^2}-\frac{1}{(2m+3)^2}-\cdots.\end{aligned}$$

Now

$$\begin{aligned}\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\cdots&=-\int\frac{dx}{x^2}+\frac{1}{2x^2}-\frac{1}{12}\frac{d}{dx}\frac{1}{x^2}+\cdots\\&=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+\cdots,\end{aligned}$$

whence, rejecting terms of the order m^{-2} and higher orders,

$$\frac{1}{(2m)^2}+\frac{1}{(2m+1)^2}+\cdots=\frac{1}{2m},$$

and

$$\log q=-\frac{1}{2m}+\frac{1}{4m}=-\frac{1}{4m}=-\log\left(1+\frac{1}{4m}\right);$$

so that

$$q=\frac{4m}{4m+1}.$$

It thus appears that the continued fraction puts in evidence an important correction, which cannot be derived with facility from the product, nor with such rigour from the series.

10. It is interesting to investigate the resulting error in calculating π from the first n factors of Wallis's product (which correspond to the nth convergent to the continued fraction), and to examine in what ratio the error is reduced by Sylvester's correction.

Let
$$v = \frac{2.2.4.4.6.6 \ldots \quad 2m}{1.3.3.5.5.7 \ldots (2m-1)},$$
$$u = \frac{2.2.4.4.6.6 \ldots \quad 2m \quad . \quad 2m}{1.3.3.5.5.7 \ldots (2m-1)(2m+1)},$$
and
$$u' = \frac{2.2.4.4.6.6 \ldots \quad 2m \quad . \quad 4m}{1.3.3.5.5.7 \ldots (2m-1)(4m+1)};$$
then
$$v = \frac{(2^2.4^2.6^2 \ldots 4m^2)^2}{2m\,(1.2.3 \ldots 2m)}.$$

Reducing this by means of the formula
$$1.2.3 \ldots x = \sqrt{(2\pi x)}\, x^x e^{-x} \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right\},$$
and rejecting terms above m^{-2}, we find
$$v = \frac{\pi}{2}\left\{1 + \frac{1}{12m} + \frac{1}{288m^2}\right\}^4 \left\{1 + \frac{1}{24m} + \frac{1}{1152m^2}\right\}^{-2}$$
$$= \frac{\pi}{2}\left\{1 + \frac{1}{4m} + \frac{1}{32m^2}\right\}.$$

Multiply by $\left(1 + \frac{1}{2m}\right)^{-1}$, viz., by $1 - \frac{1}{2m} + \frac{1}{4m^2}$, and we obtain
$$u = \frac{\pi}{2}\left\{1 - \frac{1}{4m} + \frac{5}{32m^2}\right\}.$$

If we apply the correction, and multiply v by $\left(1 + \frac{1}{4m}\right)^{-1}$, we obtain
$$u' = \frac{\pi}{2}\left\{1 + \frac{1}{32m^2}\right\},$$
so that the replacing $\frac{2m}{2m+1}$ by $\frac{4m}{4m+1}$ destroys the term in $\frac{1}{m}$.

Thus, n being large, the error in the value of π obtained from the first n terms of Wallis's formula (uncorrected) is nearly $\frac{\pi}{2n}$, and from the first n terms (corrected) is nearly $\frac{\pi}{8n^2}$; so that the error in the latter case is about the $\left(\frac{1}{4n}\right)$th part of what it is in the former. This exactly agrees with what Professor Sylvester found, the concluding

sentence of his paper being—"The approximation being thus more than fifteen and twenty-one times bettered for the fourth and fifth convergents respectively by aid of the correction."

Of course the product (or continued fraction), even when corrected, is very unsuitable for the calculation of π. The number of decimal places that the first n factors of Wallis's product afford is equal to the integer next below $\log_{10}\frac{2n}{\pi}$, viz., next below $\log_{10} n - \cdot 1961\ldots$, or, roughly, next below $\log_{10} n$; while, if the correction is applied to the nth factor, the number of places is equal to the integer next below $\log\frac{8n^2}{\pi}$, viz., next below $2\log_{10} n + \cdot 4059\ldots$, or, roughly, $2\log_{10} n$; so that the application of the correction about doubles the number of decimal places afforded by the first n terms of the product. When the correction is made, 1000 factors of the product yield but six decimal places of π; and if the correction is not made, only two.

Postscript.—*Added June* 27, 1874.—Since this paper was read, Mr. Muir has called my attention to a paper by Stern in the tenth volume of "Crelle" (1833), in which Sylvester's continued fraction for π is given. On p. 267, Stern converts the product $\frac{d_1 d_2 \ldots d_n}{e_1 e_2 \ldots e_n}$ into a continued fraction by a process very similar in principle to that used in this paper, and thence derives Sylvester's fraction from Wallis's formula. He does not, however, obtain the transformations of $\sin\frac{\pi}{n}$ and $\cos\frac{\pi}{n}$, or of the other products discussed in this paper, so that I have not thought it worth while to make any alteration in consequence.

Stern attributes Sylvester's fraction to Euler (Comm. Acad. Petropol., t. xi., p. 48 (1739); and I find that it is there obtained from the consideration of the ratio of the integrals $\int_0^1 \frac{y\,dy}{\sqrt{(1-y^2)}}$ and $\int_0^1 \frac{dy}{\sqrt{(1-y^2)}}$, which are to one another as 2 to π. In the Phil. Mag. for May, 1874, Mr. Muir has also deduced Sylvester's fraction by means of an integral.

June 11*th*, 1874.

Dr. HIRST, F.R.S., President, and subsequently Prof. CAYLEY, F.R.S., V.P., in the Chair.

The Rev. A. J. Stevens, M.A., and Mr. W. Ritchie, were elected Members of the Society; Dr. Casey was proposed for election; and Mr. G. S. Carr was admitted into the Society.

Before vacating the Chair, the President made a statement to the effect that he had much pleasure in announcing to the Members present that it was Lord Rayleigh's intention to present to the Society the sum of £1000, to assist in the publication of the Proceedings and for the purchase of Mathematical Journals.

Mr. Roberts proceeded to give a sketch of his paper "On the Parallels of Developables and of Curves of Double Curvature."

Lord Rayleigh next brought forward a "Note on the Numerical Calculation of the Roots of Fluctuating Functions."

The Secretary then read parts of papers by Mr. J. Griffiths "On a remarkable relation between the difference of two Fagnanian arcs of an ellipse of eccentricity e and that of two corresponding arcs of a hyperbola of eccentricity $\frac{1}{e}$;" by Mr. Routh, F.R.S., "On Stability of a Dynamical System with two Independent Motions," "On Rocking Stones," and "Small Oscillations to any Degree of Approximation."

The following presents were received:—

"Euclidian Geometry," by F. Cuthbertson, M.A.: from the Author.

"Journal of the Institute of Actuaries and Assurance Magazine," vol. xviii., Pt. ii., No. xciv., Jan. 1874.

"Bulletin des Sciences Mathématiques et Astronomiques," tome sixième, Mai 1874.

"Proceedings of Royal Society," vol. xxii., Nos. 150, 151.

"Demonstration géométrique de quelques théorèmes au moyen de la considération d'une rotation infiniment petite," and "Construction directe du rayon de courbure de la courbe de contour apparent d'une surface qu'on projette orthogonalement sur un plan;" par M. A. Mannheim: from the Author.

"Astronomical and Meteorological Observations made during the year 1871 at the United States Naval Observatory:" Rear-Admiral B. F. Sands, Washington, 1873.

"A new short Treatise of Algebra, with the Geometrical Construction of Equations as far as the Fourth Power or Dimension, together with a Specimen of the Nature and Algorithm of Fluxions," by John Harris, D.D.: London, 1714.

"Traité des progressions par addition ou des séries algébriques, terminé par de nouvelles vues sur la quadrature du cercle et précédé par un discours sur la nécessité d'un nouveau système de calcul," troisième edition: Paris, 1795.

The last two presented by Mr. Drach.

On the Parallel Surfaces of Developables and Curves of double Curvature. *By* Samuel Roberts, M.A.

[*Read June 11th*, 1874.]

1. In order to deal with this subject satisfactorily, I have found it desirable to consider particularly the section at infinity of the parallel surface in connection with the corresponding section of the surface of centres. I shall suppose that there is no special relation to the plane at infinity.

Firstly with respect to the latter surface of centres, we know that the section in question is made up of (α) the polar reciprocal of the section at infinity of the primitive surface taken with respect to the imaginary circle and counting three times, the order of this curve is therefore = order of tangent cone = a; of (β) another curve whose order = class of spinode torse of the primitive = c'; and of (γ) the envelope of the normals at infinity of the surface, the order of this curve is therefore $3a+c$, where c is the degree of the cuspidal curve of the primitive.

The curve (α) is cuspidal on the surface of centres, and the plane at infinity is the cuspidal tangent plane at all the points of the curve, which corresponds to radii of curvature at infinity due to successive intersections of normals in planes not coincident with the plane at infinity. The curve (β) corresponds to radii of curvature at points in general finite of the parabolic curve. The curve (γ) is due to the circumstance that the plane at infinity is a principal normal plane with regard to points of the primitive infinitely distant. The order of the surface of centres is therefore $6a+c+c'$.*

2. In the next place we must determine the correspondence of infinitely distant points of the parallel surface with reference to the primitive.

Let X, Y, Z be the coordinates of a point on the parallel surface corresponding to a point of the primitive, the coordinates of which are x, y, z. If we write U_1, U_2, U_3 for $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$, and P for the sum of

* In consequence of the peculiarity of the imaginary circle as confocal with any conic at infinity, the curves (α) and (γ) are specially related so that they touch at $2m$ points (m being the order of the primitive surface), and their common tangents are the tangents of the imaginary circle at the points of its intersection with the section of the primitive. These points are special umbilics on any surface in the sense that both the inflexional tangent lines at such a point of course meet the circle at infinity.

the squares of these functions, we have

$$X = x \pm \frac{kU_1}{\sqrt{P}},\ Y = y \pm \frac{kU_2}{\sqrt{P}},\ Z = z \pm \frac{kU_3}{\sqrt{P}}.$$

When (X, Y, Z) is a point at infinity, we may take it that X, Y, and Z are all infinite. If one of the three is finite, the point lies in the plane of the other two coordinates; and if two of them are finite, the point is the infinitely distant extremity of the other axis. These points, which are finite in number, are not essentially distinguished from the rest.

Consistently then with the above equations we may have

(α'.) (x, y, z), a finite point of the curve of contact of the focal torse proper, satisfying therefore $U = 0$, $P = 0$. In this case (X, Y, Z) is a point on the imaginary circle.

We may have

(β'.) (x, y, z), a point at infinity. The corresponding points form, as we know, the section of the primitive at infinity twice repeated.

We may also have

(γ'.) (x, y, z), a point at infinity on the curve of contact of the focal torse; these points correspond to points (X, Y, Z) whose locus is made up of the tangents common to the plane section of the primitive and the imaginary circle.

3. We can now determine the points at infinity of the cuspidal curve of the parallel surface, modulus k.

For, first of all, it is plain these must be included amongst the intersections of the section at infinity of the parallel surface, and the corresponding section of the surface of centres, since the cuspidal curve in question is the locus of points on the parallel surface, such that one of the principal radii of curvature at each of the corresponding points of the primitive is equal to the modulus k.

We must count

(α''.) The intersections of the imaginary circle with the polar reciprocal of the section of the primitive at infinity, and these points must be counted three times, because the polar reciprocal is three times on the surface of centres. These intersections are, in fact, the points of contact with the imaginary circle of the common tangents of the section of the primitive and the circle.

We also must count

(β''.) The intersections of the curve (β) of the surface of centres with the imaginary circle. These points correspond to the intersections of the curve of contact of the focal torse and the parabolic curve.

Finally we must count

(γ''.) Some of the intersections of the section of the primitive with the curve (γ) of the surface of centres, namely those which correspond to the cusps of a plane parallel of the section when the modulus is

evanescent. The number of such cusps is we know $6a+k$ for a modulus k. In the special case we have $2c$ cusps, being the cusps of the section of the primitive taken twice over. The evolute of course passes through these. It also touches the section of the primitive in $2a$ points, namely the points of contact on the section of the primitive of the tangents which are common to it and to the imaginary circle. These $2a$ points of contact taken twice over take the place of, and represent $3 \cdot 2a$ cusps of the general plane parallel of the section.

The points of the cuspidal curve at infinity consist therefore of the points of contact of the common tangents of the section of the primitive and the imaginary circle; they are to be counted three times, and are stationary points.

The points in addition are the $2c'$ points on (β) and the imaginary circle, and the $2c$ cusps of the section of the primitive twice over. The order of the curve consequently is

$$12a+2c+2c',$$

or twice the order of the surface of centres.

4. The value of the results obtained lies in the generality of their expression in terms of $3a+c$, and $3a+c'$. We may, for example, suppose that the primitive is a developable surface. In this case, the tangent cone degenerates into a planes, and c is the order of the edge of regression. Also $3a+c'=3n+\kappa$, where n is the class of the tangent cone, and K is the number of cuspidal edges, therefore $3a+c'=0$, giving $c'=-3a$. The discussion of this negative result would lead to too long a digression. It will be immediately seen that to make Plücker's equations consistent for the order 1 and class 0, we must consider linearity as indicated by the effect of four double tangents, less that of three inflexional tangents; and for the order 0 and class 1, we must consider a point as being of the nature of a summit, and endowed with a singularity equivalent to the effect of four double points less that of three cusps.

Let us now suppose that the primitive is a curve. The surface order is 0, the tangent cone is a cone enveloping the curve. In this case $3a+c=0$.

We avoid the negative value of c (but only in its becoming implicit instead of explicit), by taking the equivalent expression $3m+i$, when m is the order of the surface (in the present case, zero), and i is the number of inflexions in a plane section (in the present case, zero).

5. I proceed now to the more immediate subject of the paper.

The parallel surface of a developable, is obviously also a developable, for to each tangent plane of the primitive correspond two tangent planes of the parallel surface, touching along right lines which are the loci of the extremities of equal normal lengths along the corresponding generators of the primitive.

The section of the parallel surface at infinity consists of (1) the corresponding section of the primitive taken twice and forming a double curve on the surface; (2) the imaginary lines in which the tangent planes of the developables, also touching the imaginary circle, meet the plane at infinity. These lines are of course common tangents of the section of the primitive and the imaginary circle. The circle itself is not in general on the surface, since the curve of contact of the focal torse has degenerated into a system of points.

To any set of tangent planes of the primitive similarly directed towards a point at infinity, intersecting, in other words, in parallel lines, correspond twice the number of tangent planes of the parallel developable.

To each stationary plane of the primitive correspond two stationary planes of the parallel surface.

The edge of regression of the parallel surface is, by what precedes, a curve on the surface of centres, *i.e.*, the rectifying developable of the edge of regression of the primitive.

Since the section at infinity of this developable is the quasi-evolute of the section of the primitive in that plane, the orders of the developable and the evolute are the same, and the order of the edge of regression of the parallel surface is twice the order of the evolute.

Since the polar reciprocal of the section of the primitive with regard to the imaginary circle is just as definite as in the general case, it may be well to explain its absence in the present case. It will be remembered that the order of the surface of centres was obtained as order of evolute of plane section of primitive + order of evolute of plane section of tangent cone.* But in the present case the latter section consists of a lines, and the order of the evolute is 0.

6. In ordinary cases we only require the numbers of three singularities of a developable to determine the rest, and we have attained the numbers of four.

According to the following notation :—

r = surface order of primitive,

n = class of primitive torse = surface order of tangent cone,

m = degree of cuspidal curve or edge of regression,

a = number of stationary planes;

and (r), (n), (m), (a) being the corresponding numbers for the parallel surface, we have

$$(r) = 2(r+n),$$
$$(n) = 2n,$$
$$(m) = 2(3n+m),$$
$$(a) = 2a,$$

from which the other numbers can be computed.

* Proceedings, March 13, 1873. On Normals, &c.

It will now be observed that the characteristics of a plane section of the parallel surface are the same as those of a parallel of a plane section of the primitive in its plane.

7. The parallel of a curve in space is a tubular surface, and its theory as such has for a long time entered more or less into treatises on Geometry. Still it may be useful to regard these surfaces from the present point of view.

The points of the curve at infinity are nodal on the parallel surface. The tangents from those points to the imaginary circle also lie on the surface, and the circle itself is multiple upon it, the degree of multiplicity being equal to the rank of the primitive. An enveloping cylinder corresponds to a cylinder enveloping the parallel surface, and the latter is a parallel surface of the first cylinder to the given modulus. We thus get the characteristics of the tangent cone.

By the general formula the order of the cuspidal curve is twice the order of the surface of centres, which in the present case is the polar developable of the primitive curve. As to this, the section at infinity consists of the lines reciprocal to the infinitely distant points of the primitive with regard to the imaginary circle, and a curve whose order is the class of the curve system, or the class of the spinode torse which is in fact the torse of which the curve is the edge of regression.

8. Using the previous notation so that r is the rank of the primitive, m its degree, and n its class, we have for the parallel surface

Order	$= 2(r+m)$,
Order of tangent cone	$= 2(r+m)$,
Order of nodal curve	$= 2(r+m)^2-2r-11m-3n$,
Order of cuspidal curve	$= 2(3m+n)$,
Class	$= 2r$,
Class of node couple torse	$= r^2-r+2$ (class of n. c. torse of primitive),
Class of spinode torse	$= 2n$.

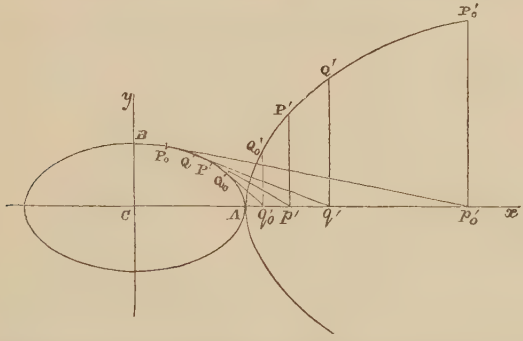
In terms of these a number of other characteristics can be determined.

In conclusion, I may remark that a parallel of a plane curve in its plane is a case of a class of limited parallels which might be investigated. Thus, relative to a curve on a surface, we can conceive a parallel curve generated by the extremities of equal normal lengths along the curve. On a tubular surface, again, any curve may be considered as parallel to the curve which is the axis of the surface. Such curves, however, may not be algebraical, and it is only in very particular cases that it might be worth while to investigate their characteristics.

On a Remarkable Relation between the Difference of two Fagnanian Arcs of an Ellipse of eccentricity e, and that of two corresponding Arcs of a Hyperbola of eccentricity $\frac{1}{e}$. *By* John Griffiths, M.A.

[*Read June* 11*th*, 1874.]

Let APQ be an ellipse whose major axis is unity, and eccentricity e; AP′Q′ a hyperbola having its transverse axis coincident both in magni-



tude and direction with the major axis of the ellipse, but whose eccentricity is $\frac{1}{e}$; also let P, Q; P_0, Q_0 be two pairs of Fagnanian points on the ellipse; P′, Q′; P_0', Q_0' two pairs of points on the hyperbola corresponding to the former in such a way that if x, ξ, x_0, ξ_0 be the abscissæ of P, Q, P_0, Q_0, on ellipse $x^2 + \frac{y^2}{1-e^2} = 1$, and x', ξ', x_0', ξ_0' those of P′, Q′, P_0', Q_0', on hyperbola $x^2 - \frac{e^2y^2}{1-e^2} = 1$, respectively, they are connected by the relations

$$xx' = \xi\xi' = x_0x_0' = \xi_0\xi_0' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

or, in other words, let P and P′, for instance, correspond so that the foot of the ordinate of P′ is the point where the tangent to the ellipse at P meets the common transverse axis of the two curves. Then I propose to show that

$$\frac{\text{arc P P}_0 - \text{arc Q Q}_0}{\text{arc P′P}_0' - \text{arc Q′Q}_0'} = e^3 x x_0 \xi \xi_0.$$

In the first place, if the abscissæ of the two pairs of points P, Q; P_0, Q_0, on the ellipse, be connected by the relations

$$\left.\begin{aligned} 1 - x^2 - \xi^2 + e^2x^2\xi^2 &= 0 \\ 1 - x_0^2 - \xi_0^2 + e^2x_0^2\xi_0^2 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2),$$

we know by Fagnani's theorem that

$$PP_0 - QQ_0 = e^2(x\xi - x_0\xi_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also, by virtue of (1) and (2), those of the corresponding points P′, Q′; P'_0, Q'_0 will be connected by the equations

$$1 - e'^2(x'^2 + \xi'^2) + e'^2 x'^2 \xi'^2 = 0,$$
$$1 - e'^2(x'^2_0 + \xi'^2_0) + e'^2 x'^2_0 \xi'^2_0 = 0,$$

if $ee' = 1$. But when this is the case, we know that the arcs of the hyperbola have the relation

$$-P'P'_0 + Q'Q'_0 = e'(x'\xi' - x'_0\xi'_0),$$

or

$$P'P'_0 - Q'Q'_0 = \frac{1}{e}\,\frac{x\xi - x_0\xi_0}{xx_0\xi\xi_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Hence, dividing (3) by (4), we have

$$\frac{\text{arc } P\,P_0 - \text{arc } Q\,Q_0}{\text{arc } P'P'_0 - \text{arc } Q'Q'_0} = e^3 xx_0\xi\xi_0.$$

It may be observed that the relation (2) between P and Q on the ellipse is equivalent to the following: viz., $pp' = ab$, where p and p' are the lengths of the perpendiculars from the centre of the curve upon the tangents at P and Q, and a, b are the principal semi-diameters.

Also, the locus of the point of intersection of the tangents at P, Q consists of an ellipse and hyperbola each confocal with the given ellipse, and passing through the vertices of the rectangle circumscribed to it at the extremities of its principal axes.

[Note by Prof. Cayley.—Viz., these are

$$X^2 \pm \frac{Y^2}{\sqrt{1-e^2}} = 1 \pm \sqrt{1-e^2}.$$

In fact, expressing y, ξ, η in terms of x, we find for the coordinates X, Y of the point of intersection

$$X = \frac{\pm\sqrt{1-e^2}\,x - \sqrt{1-x^2}\,\sqrt{1-e^2}x^2}{-1 + x^2(1 + \sqrt{1-e^2})},$$

$$\pm\frac{Y}{\sqrt{1-e^2}} = \frac{x\sqrt{1-e^2}\,x^2 - \sqrt{1-x^2}}{-1 + x^2(1 \pm \sqrt{1-e^2})},$$

whence the relation in question.

It may be added that the locus of the point of intersection of the tangents at P′, Q′ consists of the two conics

$$\frac{e^2X^2}{1 \pm \sqrt{1-e^2}} \pm \frac{e^2Y^2}{\sqrt{1-e^2}} = 1.]$$

Stability of a Dynamical System with two Independent Motions.

By E. J. Routh, F.R.S.

[*Read June 11th*, 1874.]

The equations of motion of a dynamical system performing small oscillations with two independent motions are of the form

$$\left.\begin{aligned} A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx+F\frac{d^2y}{dt^2}+G\frac{dy}{dt}+Hy=0 \\ A'\frac{d^2x}{dt^2}+B'\frac{dx}{dt}+C'x+F'\frac{d^2y}{dt^2}+G'\frac{dy}{dt}+H'y=0 \end{aligned}\right\}.$$

To solve these, we eliminate either x or y, and obtain a biquadratic of the form $$aD^4+bD^3+cD^2+dD+e=0.$$

The whole nature of the motion depends on the forms of the roots of this equation.

Rules are given in books on the theory of equations to determine whether the roots are real or imaginary; but this is not exactly what we want to know. It is often important to ascertain whether the equilibrium about which the oscillation takes place is stable or unstable. The necessary and sufficient conditions for stability are that the real roots and the real parts of the imaginary roots should be all negative. It is proposed here to investigate a method of easy application to decide this point.

Let the biquadratic be written in the form

$$\phi(x)=ax^4+bx^3+cx^2+dx+e=0.$$

Let us form the symmetrical function of the roots which is the product of the sums of the roots taken two and two. If this be called $\frac{X}{a^3}$, we find

$$\begin{aligned} X &= bcd-ad^2-eb^2 \\ &= \tfrac{1}{2}\begin{vmatrix} 2a & b & c \\ b & o & d \\ c & d & 2e \end{vmatrix}. \end{aligned}$$

Suppose we know the roots to be imaginary, say $\alpha\pm p\sqrt{-1}$, $\beta\pm q\sqrt{-1}$. Then $$\frac{X}{a^3}=4\alpha\beta\{(\alpha+\beta)^2+(p+q)^2\}\{(\alpha+\beta)^2+(p-q)^2\}.$$

Thus $\alpha\beta$ always takes the sign of $\frac{X}{a}$, and $\alpha+\beta$ always takes the sign of $-\frac{b}{a}$. Thus the signs of both α and β can be determined; and if a, b, X have the same sign, the real parts of the roots are all negative.

Suppose, next, that two of the roots are real and two imaginary. Writing $q'\sqrt{-1}$ for q, so that the roots are $\alpha \pm p\sqrt{-1}$ and $\beta \pm q'$, we find $$\frac{X}{a^3} = 4a\beta\{[(\alpha+\beta)^2+p^2-q'^2]^2+4p^2q'^2\}.$$

Just as before, $a\beta$ takes the sign of $\frac{X}{a}$, and $\alpha+\beta$ takes the sign of $-\frac{b}{a}$. Also, $\beta^2-q'^2$ takes the sign of the last term $\frac{e}{a}$ of the biquadratic. This determines whether β is numerically greater or less than q'. If, then, a, b, e, and X have the same sign, the real roots and the real parts of the imaginary roots are all negative.

Lastly, suppose the roots to be all real. Then, if all the coefficients are positive, we know, by Descartes' rule, that the roots must be all negative, and the coefficients cannot be all positive unless all the roots are negative. In this case, since X is the product of the sums of the roots taken two and two, it is clear that $\frac{X}{a}$ will be positive.

Whatever the nature of the roots may be, yet if the real roots and the real parts of the imaginary roots be negative, the biquadratic must be the product of quadratic factors all whose terms are positive. It is therefore necessary for stability that every coefficient of the biquadratic should have the same sign.

Summing up the several results which have just been proved, we conclude that *it is necessary and sufficient for stability of equilibrium that every coefficient of the biquadratic, and also* X, *should have the same sign.*

If the equation on which the motion of the system depends is of the fifth degree, we may proceed in the same way. Let the equation be $$\phi(x) = ax^5+bx^4+cx^3+dx^2+ex+f = 0.$$

Form as before the product of the sums of the roots taken two and two. If this be $\frac{X}{a^4}$, we find $$X = \begin{vmatrix} bc-ad & be-af \\ be-af & de-cf \end{vmatrix}.$$

Suppose that there are four imaginary roots $\alpha \pm p\sqrt{-1}$, $\beta \pm q\sqrt{-1}$, and one real root γ. Then γ has the sign opposite to $\frac{f}{a}$, and $a\beta$ takes the sign of X, while $2(\alpha+\beta)+\gamma = -\frac{b}{a}$. If then $\frac{f}{a}$ be positive, γ is negative. If $\frac{b}{a}$ be positive and $\phi\left(-\frac{b}{a}\right)$ negative, the root γ is numerically less than $\frac{b}{a}$, so that $\alpha+\beta$ is negative. If therefore a, b, f, X, and $-\phi\left(-\frac{b}{a}\right)$ be all positive, α, β, γ will be all negative.

Suppose that there are two imaginary roots $\alpha \pm p\sqrt{-1}$, and three real roots β, γ, δ. Then, if all the coefficients are positive, β, γ, δ are

negative, and X takes the sign opposite to α; so that if X be also positive, α, β, γ, δ will be all negative.

Suppose all the roots to be real; then, if all the coefficients be positive, the roots will be all negative, and not otherwise; and it is also clear that X, being the product of ten negative quantities, will be positive.

In both these cases, if the real roots and the real parts of the imaginary roots be negative, it is clear that $\phi\left(-\frac{b}{a}\right)$ must have the sign opposite to a.

Conversely, if all the real roots and the real parts of the imaginary roots be negative, the factors of the equation, and therefore the equation itself, must have all the coefficients of the same sign.

We therefore conclude that *it is necessary and sufficient for stability of equilibrium that every coefficient of the equation,* $-\phi\left(-\frac{b}{a}\right)$, *and also* X, *should be positive.*

When the dynamical system has three independent motions, the solution in general depends on an equation of the sixth degree. The condition of stability is, as before, that the real roots and the real parts of the imaginary roots should all be negative. This case may be discussed in the same way as the biquadratic. First, we may show that all the coefficients in the equation must be of the same sign; secondly, we may then form two determinants X, Y, which are the products of the sums of the roots taken respectively two and two together, and four and four together. It may then be shown that if all the coefficients of the equation be positive, and also X and Y both negative, the above-mentioned condition of stability is satisfied. The expressions for X and Y in terms of the coefficients of the original equations may be found without difficulty. That for X may be put into the form of a determinant of three rows,

$$\begin{vmatrix} gb & af-be & bc-ad \\ f & -d & b \\ ef-dg & bg-cf & af \end{vmatrix}.$$

That for Y may also be written in a determinant of the same kind; but the constituents are not sufficiently simple and short to render the expression of much *practical* value.

On Rocking Stones. *By* E. J. Routh, F.R.S.

[*Read June 11th*, 1874.]

The following problem seems to me interesting from its generality, and the simplicity of the results to which we are led. I have not thought it necessary to do more than indicate the general course of the demonstration.

A heavy body oscillates in three dimensions, with one degree of freedom, on a fixed rough surface of any form in such a manner that there is no rotation about the common normal.

1. Let O be the point of contact, Oz the common normal, Oy a tangent to the arc of rolling determined by the geometrical conditions of the question, OI the instantaneous axis. Then OI, Oy are conjugate diameters in the relative indicatrix.

The relative indicatrix is a conic having its centre at O and lying in the common tangent plane at O, such that the difference of the curvatures of the normal sections through any radius vector OR varies as $\frac{1}{OR^2}$.

2. Let ρ, ρ' be the radii of the normal sections through Oy, taken positively when the curvatures are in opposite directions, and let

$$\frac{1}{s} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

Measure a length $s \cdot \sin^2 y\mathrm{OI}$ along the common normal Oz, and describe a cylinder on it as diameter, the axis being parallel to OI. If the centre of gravity of the body be inside, the equilibrium is stable; if outside and above, unstable. This cylinder may therefore be called the cylinder of stability.

3. Let G be the centre of gravity, and let OG produced cut the cylinder of stability in V; then if K be the radius of gyration about OI, the length L of the simple equivalent pendulum is given by

$$\frac{K^2}{L} = \mathrm{GV} \cdot \sin^2 \mathrm{GOI}.$$

This equation may also be written in the form

$$\frac{K^2}{L} = s \cos \mathrm{G}oz \cdot \sin^2 y\mathrm{OI} - \mathrm{OG} \cdot \sin^2 \mathrm{GOI}.$$

This result may be obtained by taking moments about the instantaneous axis.

4. The motion of the upper body is the same as if the fixed surface were plane and the curvature of the upper body at the point of contact altered so that the relative indicatrix remain the same as before. This supplies an easy method of finding the oscillations in any particular case.

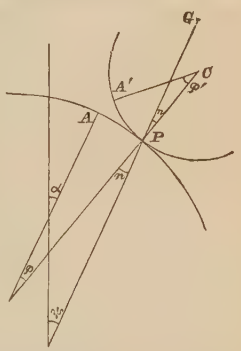
Small Oscillations to any Degree of Approximation.

By E. J. Routh, F.R.S.

[*Read June 11th*, 1874.]

In discussing the small oscillations of a dynamical system, we are usually content to reject the squares of all small quantities. It is clear, however, that in many cases the terms of the higher orders may rise in importance, and may even alter the period of the principal oscillation. It is proposed to investigate an easy method of determining the oscillation of a heavy body moving in any manner with *one independent motion* to any degree of approximation. This we shall do, first, for any small oscillation in two dimensions; and, secondly, for an oscillation in three dimensions about a fixed point.

Let P be the instantaneous centre of rotation at the time t, AP its path in space, A′P in the body. Then the motion may be constructed by making the curve A′P, fixed in the body, roll without sliding on the curve AP fixed in space. Let $AP = A'P = s$. Let G be the centre of gravity of the body $GP = r$. Let ψ be the angle GP makes with the vertical, n the angle it makes with the normal, to the curves AP, A′P at P. Let θ be the angle any straight line fixed in the body makes with a straight line fixed in space, κ the radius of gyration of the body about the centre of gravity.



Taking moments in the usual way about the instantaneous centre of rotation considered as a moving point, the equation of motion is

$$(r^2+k^2)\frac{d^2\theta}{dt^2}+rz\sin n\left(\frac{d\theta}{dt}\right)^2=gr\sin\psi.$$

The method of proceeding is as follows:—To solve the equation, we must expand each coefficient by Taylor's theorem in powers of θ, which is to be so chosen as to vanish in the position of equilibrium. To do this we require the successive differentials of the coefficients to any order expressed in terms of the *initial* values only of ψ, n, and r. We find by inspection of the figure

$$\frac{d\psi}{d\theta}=1-z\frac{\cos n}{r},$$

$$\frac{dn}{d\theta}=z\left(\frac{\cos n}{r}-\frac{1}{\rho'}\right),$$

$$\frac{dr}{d\theta}=z\sin n,$$

$$\frac{1}{z} = \frac{1}{\rho} + \frac{1}{\rho'}, \quad z = \frac{ds}{d\theta},$$

where ρ' is the radius of curvature of A′P, and ρ of AP at P, and are both supposed to be given functions of s. All that is necessary is to differentiate each coefficient in the differential equation, and to substitute in the result from these equations the values of $\frac{d\psi}{d\theta}$, $\frac{dn}{d\theta}$, $\frac{dr}{d\theta}$, as they enter. In equilibrium, GP is vertical; hence the initial value of ψ is zero. Let the initial or equilibrium values of n and r be α and h. These equations we shall refer to as "the subsidiary equations."

To solve the equation to the first order.—We have

$$(h^2+k^2)\frac{d^2\theta}{dt^2} = gr\sin\psi.$$

We have only to calculate $r\sin\psi$ to the first power of θ,

$$\begin{aligned} r\sin\psi &= \frac{d}{d\theta}(r\sin\psi)\,\theta \\ &= \left\{\sin n\sin\psi + r\cos\psi\left(\frac{1}{z} - \frac{\cos n}{r}\right)\right\} z\theta \\ &= \{r\cos\psi - z\cos(\psi - n)\}\,\theta. \end{aligned}$$

Hence the equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{z\cos\alpha - h}{h^2+k^2}\theta = 0.$$

To solve the equation to the second order.—We have just found

$$\frac{d}{d\theta}(r\sin\psi) = r\cos\psi - z\cos(\psi - n).$$

We must differentiate this, and retain only the terms which do not vanish when $\psi = 0$. We get

$$\frac{d^2}{d\theta^2}(r\sin\psi) = z^2\left\{z\cos\alpha\frac{d\frac{1}{z}}{ds} + \frac{\sin 2\alpha}{h} - \frac{\sin\alpha}{\rho'}\right\}.$$

Hence the equation of motion to the second order is

$$\{h^2+k^2+2hz\sin\alpha\theta\}\frac{d^2\theta}{dt} + hz\sin\alpha\left(\frac{d\theta}{dt}\right)^2$$

$$= -(z\cos\alpha - h)\,g\theta + gz^2\left\{z\cos\alpha\frac{d\frac{1}{z}}{ds} + \frac{\sin 2\alpha}{h} - \frac{\sin\alpha}{\rho'}\right\}\frac{\theta^2}{2}.$$

This is the same as $\frac{d^2\theta}{dt^2} + a^2\theta = -b^2\left(\frac{d\theta}{dt}\right)^2 + c\theta^2$,

where

$$a^2 = \frac{z\cos\alpha - h}{k^2+h^2}g,$$

$$b^2 = \frac{hz\sin\alpha}{k^2+h^2},$$

$$c = 2a^2b^2 + \tfrac{1}{2}g\frac{z^2}{k^2+h^2}\left\{ z\cos\alpha\,\frac{d\frac{1}{z}}{ds} + \frac{\sin 2\alpha}{h} - \frac{\sin\alpha}{\rho'}\right\}.$$

Supposing a not to vanish, we find

$$\theta = \mathrm{A}\sin(at+\mathrm{B}) + \frac{c-a^2b^2}{2a^2}\mathrm{A}^2 + \frac{c+a^2b^2}{6a^2}\mathrm{A}^2\cos 2(at+\mathrm{B}),$$

so that the first approximation is substantially correct unless a be small, *i.e.* unless the equilibrium be nearly neutral. The effect of the small terms is to make the extent of the oscillation on the lower side of the position of equilibrium greater than that on the upper side.

To solve the equation to the third order.—We have now to differentiate again the expressions obtained in the last approximation. The process is very easy, but the result is long. If we suppose that $n=0$ in the position of equilibrium, the equation takes the form

$$\frac{d^2\theta}{dt^2} + a^2\theta = -e^2\theta\left(\frac{d\theta}{dt}\right)^2 + f\theta^2 + f'\theta^3,$$

where e^2, f, f' are functions of h, ρ, ρ' obtained by the above process. It is obvious that one effect of these additional terms is to alter the period of the oscillation. The principal term is now

$$\theta = \mathrm{A}\sin(a't+\mathrm{B}),$$

where

$$a'^2 = g\frac{z-h}{h^2+k^2} + ghz^2\left(\frac{1}{h}-\frac{1}{\rho'}\right)\frac{z-h}{(h^2+k^2)^2}\,\tfrac{5}{4}\mathrm{A}^2$$

$$-g\left\{2z-h-\frac{z^2}{\rho'}+z\left(1-\frac{2z}{h}+\frac{z}{\rho'}\right)^2-\frac{z}{2}\frac{d^2z^2}{ds^2}\right\}\frac{1}{h^2+k^2}\cdot\frac{\mathrm{A}^2}{8}.$$

When other forces besides gravity act on the body, the problem may be treated in the same way. Thus, suppose the body acted on by a central force which passes through the centre of gravity and some point fixed in space. A very slight modification of the subsidiary equations already written down will enable us to determine the oscillations to any order.

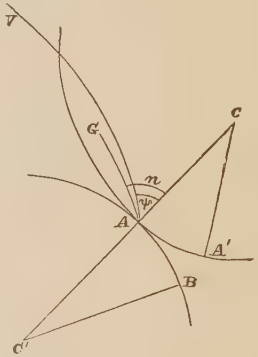
When the oscillation takes place in three dimensions about a fixed point with only one independent motion, we may represent the geometrical constraints by supposing the body to be attached to a cone without inertia which is constrained to roll on another cone fixed in space. These cones are the surfaces generated in the body and in space by the instantaneous axis. The problem is now reduced to the following simpler form:

To determine to any degree of approximation the oscillations of a heavy conical body on a fixed rough conical surface, the vertices being coincident.

Let I be the moment of inertia of the cone about the instantaneous axis, Ω the angular velocity of the body, N the moment of the forces

about the instantaneous axis. Then, by differentiating the equation of Vis Viva, we have $$I\frac{d\Omega}{dt}+\tfrac{1}{2}\frac{dI}{dt}\Omega=N.$$

Let the vertex O of the cones be the centre of a sphere of unit radius. Let the instantaneous axis cut this sphere in A, the line joining O to the centre of gravity cut it in G, and the vertical OV cut it in V. Let OC, OC′ be axes of right circular cones osculating the given cones along the instantaneous axis. Let OA′ be the generating line of the moving cone which will coincide with OB and be the instantaneous axis at time $t+dt$. Let the arcs CA, C′A be ρ, ρ'. Let the angle AOA′ be ds; then we easily find $$\Omega=\frac{\sin(\rho+\rho')}{\sin\rho\,\sin\rho'}\frac{ds}{dt}.$$



Let h be the distance of the centre of gravity from the vertex. Let the arcs $GA=r$, $VA=z$, $\angle GAC=n$, $\angle VAC=\psi$. Then we have $N=-gh\sin r\sin z\sin(n-\psi)$. In equilibrium this vanishes when $n=\psi$.

To form the equation of motion to any degree of approximation, we must expand this in powers of s by Taylor's theorem. To accomplish this, we must, as before, find $\frac{dr}{ds}, \frac{dz}{ds}$, &c., in terms of r, n, &c., so that each differential coefficient may be expressed only in terms of the initial values of the quantities.

These subsidiary equations are easily seen to be

$$\frac{dr}{ds}=\sin n, \qquad \frac{dz}{ds}=\sin\psi,$$

$$\frac{dn}{ds}=\cot r\cos n-\cot\rho,$$

$$\frac{d\psi}{ds}=\cot z\cos\psi+\cot\rho',$$

where $CA=\rho$, $C'A=\rho'$.

To solve the equation to the first order.—The equation is

$$I\frac{\sin(\rho+\rho')}{\sin\rho\,\sin\rho'}\frac{d^2s}{dt^2}=-gh\sin r\sin z\sin(n-\psi).$$

To expand the right side in powers of s, we differentiate it and substitute from the above subsidiary equations for all differential coefficients as they arise. In equilibrium $n=\psi$; hence, as we have to substitute for all the letters their initial values, we need only differentiate the last term. We get therefore

$$N=-gh\sin r\sin z\cos(n-\psi)\{(\cot r-\cot z)\cos n-(\cot\rho+\cot\rho')\}s.$$

Hence $$I\frac{d^2s}{dt^2} = -gh\left\{\sin(z-r)\cos n\frac{\sin\rho\sin\rho'}{\sin(\rho+\rho')} - \sin r\sin z\right\}s;$$

therefore, if L be the length of the equivalent pendulum,

$$\frac{I}{hL} = \sin(z-r)\cos n\frac{\sin\rho\sin\rho'}{\sin(\rho+\rho')} - \sin r\sin z.$$

For the next approximation, let us suppose the rolling body to be a right cone on the summit of another right cone. We then require to include the *cubes* of small quantities. By an easy process of differentiation we find the equation becomes

$$\frac{I}{gh}\frac{d^2s}{dt^2} = -\frac{\sin^2\rho\sin z\sin(\rho'+z)}{\sin(\rho+\rho')}s + P.\frac{\sin\rho\sin(\rho'+z)}{\sin^2 z\sin^2\rho'\sin(\rho+\rho')}\cdot\frac{s^3}{6},$$

where $$P = \sin\rho\sin^2(\rho'+z) + \sin\rho\sin^2\rho' - 3\sin(\rho'+z)\cos\rho\sin\rho'\sin z.$$

The effect of this latter term on the period of the principal term may be easily found.

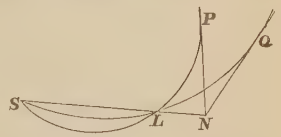
Inversion, with special reference to the Inversion of an Anchor Ring or Torus. *By* H. M. Taylor, M.A.

[*Read April 9th*, 1874.]

We premise that a straight line inverts into a circle passing through the pole, and *vice versâ*; that a circle inverts into a circle, the two circles being subcontrary sections of a cone of the second degree passing through the pole; and that the angles between lines and surfaces at their points of intersection are the same as the angles between the inverse lines and surfaces at the inverse points.

A normal is a straight line cutting a curve or a surface at right angles; it will therefore invert into a circle through the pole cutting the inverse curve or surface at the inverse point at right angles. Such a circle we will call a normal circle.

We will now prove that, if two normals at any two points of a surface intersect and be equal, the normals at the inverse points of the inverse surface also will intersect and be equal.



Let S be the pole, and let PN, QN, the normals to a surface at P, Q, be equal and intersect in N.

If we draw a circle to touch PN at P and pass through S, this will cut SN in a point L such that $NL.NS = NP^2$; and because NP, NQ are equal, the circle touching QN at Q and passing through S will pass

through the same point L. These circles invert into the normals of the inverse surface, which must therefore intersect in the inverse point of L.

Also, in the inverse figure, L and N merely interchange; hence the normals in the inverse figure are equal.

We may state this proposition in other words, thus: If a sphere be described to touch any surface at a number of points, the normals to the inverse surface at the inverse points will all intersect and be equal. This is seen at once by noticing that the touching sphere inverts into a touching sphere.

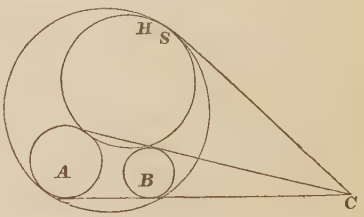
If P, Q, two such points, be very near together on a surface, the line PQ is an element of a line of curvature on the surface. If P′, Q′ be the inverse points of P, Q, then P′Q′ will be an element of a line of curvature on the inverse surface. Hence the known theorem, lines of curvature invert into lines of curvature.

If we draw the four common tangents to two given circles, the intersection of the two external tangents, the intersection of the two internal tangents, and the two centres, are collinear. The other four intersections lie on a circle whose centre is collinear with the four points. By inversion we see that, if we take any two given circles, and draw through a fixed point the four circles which touch each of the given circles, the intersection of the two whose contacts are of opposite kinds, and the intersection of the two whose contacts are of the same kind, lie on the circle through the fixed point orthogonal to the given circles; also that the other four intersections lie on a circle orthogonal to the orthogonal circle.

The locus of a point, the inverses with respect to which of two given circles are two equal circles, is the circle with respect to which the two given circles are inverses of each other.

An analytical proof of this proposition was given in this paper as first presented to the Mathematical Society; but the following proof is more in accordance with the general character of the paper:

If one circle be drawn touching each of two given circles A and B externally, and another touching each internally, these two circles in general will intersect in two points, S and H say, which are collinear with the external centre of similitude of A and B. If the figure be inverted from S as pole, the inverse of H will be the external centre of similitude of the inverses of A and B. If these inverse circles are equal, the inverse of H must go off to infinity, or, in other words, S



and H must coincide; and therefore CS^2 must be equal to the product of the tangents from C to A and B.

This proves the proposition for the locus circle whose centre is the *external* centre of similitude. The proof can easily be adapted to the circle whose centre is the *internal* centre of similitude.

If the two given circles intersect, the two locus circles are real and pass through the points of intersection. If the two circles do not intersect, one only of the locus circles is real. The two given circles and the two locus circles have the same radical axis in all cases.

It can easily be deduced, from the results already obtained, that a pair of circles and a locus circle invert, with respect to any pole in their plane, into a pair of circles and a locus circle.

It is clear that the above propositions hold also for spheres, thus: The locus of points, the inverses with respect to which of two given spheres are two equal spheres, is a sphere (the locus sphere) with respect to which the two given spheres are each other's inverses; and if we invert two spheres and their locus sphere, we obtain two other spheres and their locus sphere.

It follows that it is possible to invert, in plane geometry, any three circles into three equal circles; and, in solid, any four spheres into four equal spheres.

Any two circles on a sphere can be inverted into two equal circles in a plane through the pole by taking the pole on a definite circle on the sphere.

This can be seen in the following manner: Draw through the two circles two spheres orthogonal to the given sphere. The intersection of the sphere with respect to which these two spheres are each other's inverses with the given sphere will be the locus of the pole required.

From this we can see that any three circles on a sphere can be inverted into three equal circles in a plane through the pole, by taking one of two definite points on the sphere for pole.

An anchor ring is the surface generated by the revolution of a circle about a straight line in its plane.

This straight line is called the *axis* of the anchor ring.

Any plane through the axis cuts the surface in two equal circles of constant radius. Such a plane or section is called *axial.*

Any plane at right angles to the axis cuts the surface in two concentric circles. Such a plane or section is called *transverse.*

All the normals of the anchor ring intersect the axis, and also the circle which is the locus of the centre of the generating circle of the ring. The plane of this last circle is called the principal plane of the ring. A circular section of either system is a line of curvature.

There is a third system of circular sections of the anchor ring. It can be shown that a plane which has double contact with the ring cuts it in two equal circles which intersect at the points of contact of the plane. We will call these the *inclined* circular sections.

We can therefore make the statement that any plane which cuts the ring in a circle cuts it also in a second circle.

The same is true of spheres, as we shall afterwards prove.

The inverse of a surface which has circular lines of curvature will invert into one which has the same property.

The anchor ring has all its lines of curvature circles; therefore its inverse with respect to any pole must have the same property.

A surface which has all its lines of curvature circles is in this paper called a *cyclide.**

We may remark that the inverse surfaces of all cones, cylinders, and surfaces of revolution must have one set of their lines of curvature circles. The inverses of the right cone and circular cylinder must have all their lines of curvature circles; but as the right cone and right circular cylinder are special cases of the anchor ring, we will restrict ourselves to the anchor ring at present.

Any sphere through one circular section of an anchor ring must meet the ring in another circular section.

If the first section be *transverse*, then the sphere must have its centre in the axis, and therefore will cut the ring in another transverse section.

If the first section be *axial*, then the sphere must have its centre in the principal plane of the ring, and therefore will cut the ring in another axial section.

If the first section be *inclined*, we see that the centre of the sphere must lie on a definite straight line at right angles to the section; and, no matter what point we take in it for the centre of the sphere, we can easily show that there is a second inclined section through which the sphere must pass.

The proposition is therefore true that a sphere through *any* circular section must pass through a second.

Any plane which cuts the anchor ring in a circle, cuts it also in another circle.

Therefore any sphere or plane which cuts the cyclide in one circle, cuts it also in another circle.

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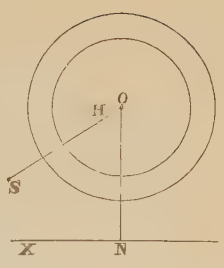
* See note at the end of the paper by Prof. Cayley.



Suppose the plane of the paper to be the principal plane of the anchor ring, O its centre. Let S be any point not in the principal plane which we will take for the pole of inversion.

Now all the spheres which pass through S and cut the anchor ring in axial circles, must pass through a point H in SO such that OH . OS is equal to the square of the tangent from O to the ring. All these spheres must have their centres on NX, the intersection of the principal plane with the plane bisecting SH at right angles; and N, the foot of the perpendicular from O on NX, will be the centre of the section of any one of these spheres by the plane SON; in other words, all these spheres have a common circular section, the circle whose centre is N and radius NS or NH.

Therefore, in the cyclide, all the planes which give circular sections of one system pass through a straight line (which will be in the plane SNO, and at right angles to SN).

Because N is the centre of this circle, and OH . OS is equal to the square of the tangent from O to it, the axis of the ring is the common radical axis of this circle and the two axial circles in the plane SON. Any circle, therefore, which cuts these axial circles orthogonally will also cut the third circle orthogonally.

Now the orthogonal sphere of the anchor ring through S will invert into a plane cutting the cyclide orthogonally at every point, which we will call one of the principal planes of the cyclide. The straight line through which we have shown the planes of one set of the circular sections must pass will therefore be at right angles to this plane.

Now we have seen that one sphere can be described through S to cut the ring orthogonally, and that its section by the axial plane through S will give a circle passing through S and cutting the two axial circles in that plane orthogonally. Now if we describe two circles to touch each of the equal circles symmetrically and pass through S, they must also pass through K, where K is such that the axis bisects SK at right angles. The inverse, therefore, of K will be the external centre of similitude of the circles, the inverses of the equal circles. Therefore all the planes of the circular sections of one system of the cyclide pass through the centre of similitude of the two circular sections of the principal perpendicular plane.

All spheres which cut the ring in one of the transverse circular sections, and pass through S, must also pass through K. Furthermore, since all their centres lie on the axis, they must have a common circular section (the circle on SK as diameter, coaxial with the anchor ring).

In the cyclide, therefore, all planes which cut the surface in one of the second set of circular sections must cut it also in a second circle, and all these planes must pass through the straight line the inverse of the circle on SK as diameter, coaxial with the ring.

It is also true that a plane or sphere passing through one of the circles which are the inverses of the inclined sections of the anchor ring, which are not lines of curvature, passes through another circular section of the same kind.

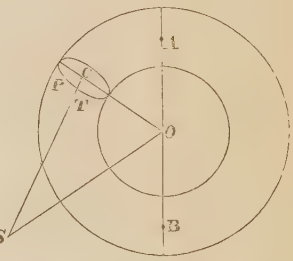
In the case of the anchor ring, it is always possible to draw through a given point a plane and a sphere, orthogonal at once to the ring and to each other, each of which intersects the ring in two circles. In the cyclide, therefore, through any point, it is possible to draw two spheres which cut the cyclide and each other orthogonally. If we invert the cyclide with respect to any point, these two spheres through the point will invert into two planes which will be the principal planes of the cyclide so obtained. They will be at right angles to each other and to the surface which they cut in two pairs of circles; and it may be shown that this second inverse is exactly of the same nature—*i. e.*, is generated in the same way—as the first inverse, or that all the cyclides may be obtained from the anchor ring by one inversion.

It is possible to draw through any point two spheres which will touch the anchor ring along circles. Every cyclide, therefore, must have a pair of tangent planes which touch the surface along circles.

Let C be the centre of an axial circle, and P any point on it; the normal circle to the anchor ring at P lies in the plane SCP, and must therefore cut CS again in some point T such that

$$CT \cdot CS = CP^2.$$

Therefore T is a fixed point for all positions of P on the axial circle.

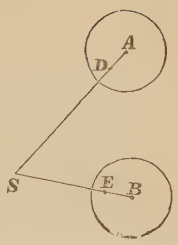


In the cyclide, therefore, all the normals along the inverse circle will pass through the inverse of T; and all the normals will intersect the curve the inverse of the locus of T.

Now T must be a point on the sphere through S, orthogonal to the anchor ring, since $CT \cdot CS = CP^2$; and it also lies on the cone of the second degree, vertex S, and base the circle the locus of C. The inverse of the locus of T is therefore a plane conic in the principal section of the cyclide.

The nature of this conic can be determined in the following manner: The extremities of its axis must lie in the lines SA, SB, where A and B are the centres of the axial sections whose plane passes through S; and they will be the inverse points of D and E in which

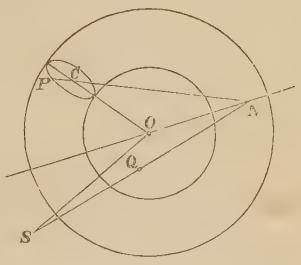
the circle through S orthogonal to these axial circles meets SA, SB.



If D, E be on the same side of S on the cone, the conic must be an ellipse, and so on.

Whence we see that the conic will be an ellipse or hyperbola according as S is outside or inside both of the circles, *or* outside only one of the circles; *i. e.*, according as S is without or within the substance of the ring.

The normal to the anchor ring at P cuts the axis in some point N.



The normal circle at P must therefore cut NS in some point Q such that

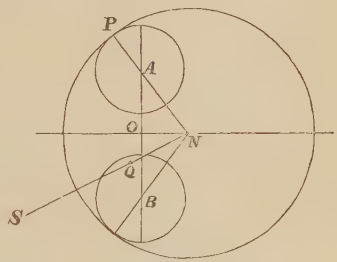
$$NQ \,.\, NS = NP^2.$$

Therefore Q is a fixed point for all positions of P on a transverse circle.

In the cyclide all the normals along one of the circular sections pass through a fixed point, and all the normals pass through a fixed plane curve the inverse of the locus of Q, which is clearly a plane curve.

We may consider the locus of Q as obtained by this same property from points P on the axial circles in the axial plane through S.

The circle whose centre is N and radius NP must touch not only the circle (centre A), but the equal circle (centre B), and Q will invert into the centre of the circle inverse of the circle (centre N).



All these circles touch two given equal circles, the normals at the points of contact intersecting in a straight line.

In the inverse figure, the circles will touch two given circles, and therefore the normals at the points of contact will intersect on a conic which will be an ellipse or hyperbola according as the pole of inversion is inside only one, *or* outside or inside both of the equal circles; *i. e.*, according as the pole is within or without the substance of the anchor ring. Of course both of these conics are parabolas, if the pole be on the surface of the ring.

We have proved that all the normals of a cyclide pass through two plane conics in planes at right angles; and since the inverse of a cyclide with respect to any pole is a cyclide, it follows that all the normal circles of a cyclide drawn through a given point pass through

two spherical curves, which would invert into plane conics; each curve being the intersection of a cone of the second degree whose vertex is the point and a sphere drawn through the point, the two spheres also being orthogonal to one another.

If the point be one such that we can invert the cyclide with respect to it into an anchor ring, these two curves are two circles in planes at right angles, one of the circles passing through the point.

Suppose a system of anchor rings generated by a series of concentric circles revolving round a fixed straight line in their plane; they of course form a series of parallel surfaces. Their orthogonal surfaces are a series of planes passing through the common axis of the rings, and a series of right circular cones coaxial with the rings and passing through the circle traced out by the revolution of the centre of the revolving circle.

If we invert the system of surfaces, we get a second system which cut each other along lines of curvature. They will be a series of cyclides the inverses of the anchor rings, a second series of cyclides the inverses of the cones, and a series of spheres the inverses of the planes.

The cyclides have been investigated before; but I do not know that they have been examined from this point of view, as the inverse figures of the anchor ring, many of whose geometrical properties are as easily seen as those of the circle. Some of the properties I have given have been given by Maxwell, Quarterly Journal of Mathematics, vol. ix., where excellent stereoscopic views of four species of cyclides are given; and by Cayley in the same Journal, vol. xii.; and in a paper in the Phil. Trans., vol. clxi., by Casey.

[Note by Prof. Cayley.—The inverse of the anchor ring (in the foregoing paper called the cyclide) is in fact the general binodal cyclide or binodal bicircular quartic; viz., assuming it to be a cyclide (bicircular quartic), to see that it is binodal, it need only be observed that the anchor ring is binodal (has two real or imaginary conic points, viz., these are the intersections of the circles in the several axial planes); and to see that it is the general binodal cyclide, we have only to count the constants; viz., general cyclide or surface

$$(x^2+y^2+z^2)^2+(x^2+y^2+z^2)(\alpha x+\beta y+\gamma z) + (a, b, c, d, f, g, h, l, m, n)(x, y, z, 1)^2 = 0$$

contains 13 constants, and therefore binodal cyclide $13-2$, $=11$ constants. But anchor ring, irrespective of position, contains 2 constants; centre of inversion, taken in given axial plane, has 2 constants; radius of inversion, 1 constant; in all $2+2+1$, $=5$ constants; or taking the inverse surface in an arbitrary position, the number of constants is $5+6$, $=11$.]

Determination of the Form of the Dome of Uniform Stress.

By C. W. Merrifield, F.R.S.

[*Read April 9th*, 1874.]

The general question of the equilibrium figure of a thin dome is indeterminate, even when the law of thickness or density is given, and it thus differs from the question of the arch, by requiring the assumption of a further condition, in order to render its form determinate.

I have found that a very remarkable simplification occurs in the analysis, when we introduce simultaneously into the general equations the two conditions:

(1) That the thrust along a meridian shall equal the thrust along the parallel per unit of area at every point;

(2) That the normal thickness shall vary in such a manner that the area under compression shall be proportional to the thrust.

These seem to be the conditions necessary to the economical use of building materials of homogeneous character, for the maximum stress is evidently least when the stress is equally distributed throughout the whole of the material, and throughout all the directions in which it is capable of being equalized.

When the thickness is finite, instead of being capable of representation by means of density, the stress throughout a given stone (or *voussoir*) is not uniform, or capable of being made so, in consequence of there being no applied pressure normal to the external or internal surfaces of the dome. Still, the distribution of stress supposed (namely, its equal distribution in the tangent plane) is the best possible.

I refer to Rankine's "Applied Mechanics," Art. 234 (Third Edition, p. 265), for the general equations of tension or stress.

Measuring x vertically downwards, and taking y for the horizontal radius at any level, also making P_x the total weight above the level of the horizontal joint whose abscissa is x, and writing $\frac{dx}{ds} = \sin i$, we have, for the total thrust along the meridians at any given level,

$$P_x \frac{ds}{dx} = P_x \operatorname{cosec} i.$$

Per unit of length along the parallel, therefore, the thrust in the direction of the meridians is $\frac{1}{2\pi y} P_x \frac{ds}{dx}$.

Again, the thrust along the parallel for each unit of length on the meridian is $\frac{1}{2\pi} \frac{d}{ds} (P_x \operatorname{cotan} i) = \frac{1}{2\pi} \frac{d}{ds} \left(P_x \frac{dy}{dx} \right)$.

The most economical use of homogeneous material would seem to be to make these thrusts equal, giving

$$P_x \frac{1}{y}\frac{ds}{dx} = \frac{d}{ds}\left(P_x \frac{dy}{dx}\right) = \frac{dP_x}{ds}\frac{dy}{dx} + P_x \frac{dx}{ds}\frac{d^2y}{dx^2},$$

or

$$P_x \frac{ds}{dx}\left\{\frac{1}{y} - \frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2\right\} = \frac{dP_x}{ds}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If the stress along a parallel be λ times the stress along a meridian, then we have

$$P_x \frac{ds}{dx}\left\{\frac{\lambda}{y} - \frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2\right\} = \frac{dP_x}{ds}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In the case of a dome constructed of stratified stone, the most economical use of material may be to give λ some value different from unity.

If t be the normal thickness of the ring at any level,—this thickness being small, so that (constants apart) it may be taken as the equivalent of density or strength,—then

$$dP_x = 2\pi wty \,.\, ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

is the weight of a narrow ring of the dome.

The bearing surface normal to the dome at the same level is $2\pi ty$. If now we suppose that the thrust is proportional to the bearing surface, so that the stress is constant at all levels, we have

$$P_x \operatorname{cosec} i = P_x \frac{ds}{dx} = 2\pi wkty \quad \ldots\ldots\ldots\ldots\ldots (4),$$

w being a constant depending upon the weight of the materials used, and k a coefficient depending upon the strength of the dome; λk in fact the linear representative of the load per unit of area along the parallels, and k along the meridians.

Substituting in (1) and (2) the values of $P_x \frac{ds}{dx}$ and $\frac{dP_x}{ds}$ given by (3) and (4), $2\pi wty$ divides out, and (2) becomes

$$k\left\{\frac{\lambda}{y} - \frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2\right\} = \frac{dy}{dx},$$

or

$$\lambda k = y\frac{dy}{dx} + \frac{ky\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

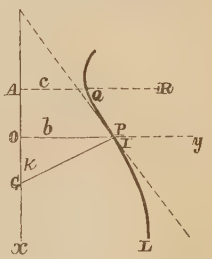
which is the differential equation of the profile.

In this equation $y\frac{dy}{dx}$ is the subnormal at any point; and if we make $\frac{d^2y}{dx^2} = 0$, we have $\lambda k = y\frac{dy}{dx}$,

that is to say, λk is the subnormal at the point of inflexion.

Again, if $\frac{dy}{dx}=0$ when $y=c$, $\lambda=c\frac{d^2y}{dx^2}$, or the radius of curvature at that point is $\frac{c}{\lambda}$. If $\lambda=1$, the radius of curvature, when y is a minimum, is equal and opposite to the distance from the axis.

Supposing the coordinates to be so taken that, when $x=0$, $y=b$, and $\frac{d^2y}{dx^2}=0$, then the point of inflexion will be on the axis of y, and the subnormal will be λk. Moreover, above this level ($y<b$) $y\frac{d^2y}{dx^2}$ is positive, and below it ($y>b$) $y\frac{d^2y}{dx^2}$ is negative. Above the point of inflexion, therefore, the curve is convex to the axis, and below it concave. The general form of the profile of the dome is therefore like the sketch.



The branch PL is parabolic. When $\lambda=1$, RQ = QA.

Equations (3) and (4) give

$$\frac{dP_x}{P_x}=\frac{1}{k}\frac{ds^2}{dx}=\frac{1}{k}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}dx \quad \ldots\ldots\ldots\ldots (6).$$

Now, multiplying (5) by $\frac{1}{y}\frac{dy}{dx}$ and transposing, we have

$$\left(\frac{dy}{dx}\right)^2=\frac{k}{y}\frac{dy}{dx}-\frac{k\frac{dy}{dx}\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}.$$

Substituting this in (6), we have

$$\frac{1}{P_x}\frac{dP_x}{dx}=\frac{1}{k}+\frac{\lambda}{y}\frac{dy}{dx}-\frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}.$$

Integrating this, we obtain

$$\log P_x+C=\frac{x}{k}+\lambda\log y-\log\left(\frac{ds}{dx}\right) \quad \ldots\ldots\ldots\ldots (7).$$

Supposing a lantern weighing W to be placed at the level of the inflexional line, we get

$$x=0,\quad y=b,\quad \frac{ds}{dx}=\sqrt{1+\frac{\lambda^2k^2}{b^2}},$$

and

$$\log W+C=\lambda\log b-\log\sqrt{1+\frac{^2k}{b^2}};$$

therefore $$\log \frac{P_x}{W} = \frac{x}{k} + \lambda \log \frac{y}{b} - \log \left(\frac{\frac{ds}{dx}}{\sqrt{1+\frac{\lambda^2 k^2}{b^2}}} \right),$$

or $$\frac{P_x}{W} = \sqrt{1+\frac{\lambda^2 k^2}{b^2}} \left(\frac{y}{b}\right)^{\lambda} \epsilon^{\frac{x}{k}} \frac{dx}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

which gives P_x. Then t is easily obtained from (4).

If we take the origin so that $x=0$ makes y a minimum, and suppose the lantern at that level to weigh L, we have

$$y=c \quad \text{and} \quad \frac{dy}{dx}=0, \quad \text{also} \quad \log L + C = \lambda \log c;$$

therefore $$\log \frac{P_x}{L} = \frac{x}{k} + \lambda \log \frac{y}{c} - \log \left(\frac{ds}{dx}\right),$$

or $$\frac{P_x}{L} = \epsilon^{\frac{x}{k}} \left(\frac{y}{c}\right)^{\lambda} \frac{dx}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

but x is now different from what it was before. Now, when $y=c$ and $\frac{ds}{dx}=1$, $P_x = 2\pi wktc$. Therefore at this level

$$2\pi wktc = L, \quad \text{or} \quad t = \frac{L}{2\pi wkc},$$

as it ought to be, for k is the height of the lantern at this level, supposing it to be made of the same homogeneous material as the dome.

Equation (5) may also be written as

$$\lambda k \frac{dx}{dy} = y - \frac{ky\frac{d^2x}{dy^2}}{1+\left(\frac{dx}{dy}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

It does not appear that it can be solved. Its reduction, by making $p = \frac{dy}{dx}$, which gives $$\lambda k = py + \frac{kpy}{1+p^2}\frac{dp}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

does not help us, unless in the case where $k=\infty$. In this case (which is unmechanical) we have also, from (5),

$$\lambda = y\frac{d^2y}{dx^2}\left(\frac{dx}{ds}\right)^2,$$

or $$-\lambda\rho = y\frac{ds}{dx}, \quad \text{the normal.}$$

When $\lambda=1$, this is Ampère's surface, generated by a catenary revolving about its directrix. When $\lambda=\frac{1}{2}$, the surface is generated by a parabola revolving about its directrix.

Reverting to the general form of the curve, it seems a little puzzling at first sight to see what becomes of it above *the neck*, and how it fulfils the conditions of equilibrium. We must, however, remember that we have assumed a pressure tangential to the meridians. On this assumption, there is nothing to prevent an overhang at the top of the dome.

The equations are satisfied by the assumption $\frac{dy}{dx} = -\infty$, if we assume that $\frac{d^2y}{dx^2} = +\infty$, even though y remains finite. In that case (4) gives $P_x = 0$, and (3) gives $\frac{dP_x}{ds} = 0$ also. That is to say, the substance of the dome thins out excessively at what we may call *the lip*.

Beyond that I have not sought to enquire what takes place. It can have no mechanical interest.

The supposition $\lambda = 0$, that is to say, that the stress along parallels vanishes, while that along meridians is constant, gives for the profile the catenary of equal stress, the axis of revolution being its asymptote, not its axis of figure. But, on determining t, we find that this solution is nugatory.

If we make $$y = c + \frac{\lambda}{2c}x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots,$$

we have
$$A_3 = -\frac{\lambda}{6ck},$$
$$A_4 = \frac{\lambda}{24c^3k^2}(c^2 + 2\lambda^2k^2 - \lambda k^2),$$
$$A_5 = -\frac{\lambda}{120c^3k^3}(c^2 + 17\lambda^2k^2 - 5\lambda k^2).$$

If $\lambda = 1$, this gives

$$y = c + \frac{x^2}{2c} - \frac{x^3}{6ck} + \frac{(c^2+k^2)x^4}{24c^3k^2} - \frac{(c^2+12k^2)x^5}{120c^3k^3} + \dots,$$

$$\frac{dy}{dx} = \frac{x}{c} - \frac{x^2}{2ck} + \frac{(c^2+k^2)x^3}{6c^3k^2} - \frac{(c^2+12k^2)x^4}{24c^3k^3} + \dots,$$

$$\frac{d^2y}{dx^2} = \frac{1}{c} - \frac{x}{ck} + \frac{(c^2+k^2)x^2}{2c^3k^2} - \frac{(c^2+12k^2)x^3}{6c^3k^3} + \dots.$$

This last is positive for $x = 0$, and negative for $x = k$; that is to say, when $\lambda = 1$, the distance along the axis of x for the points at which $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ is less than k, unless $2c^2 > 9k^2$, or unless $k < \frac{\sqrt{2}}{3}c$. This expansion assumes the origin to be so taken that $\frac{dy}{dx}$ vanishes with x, a supposition which simplifies the coefficients, but

which is by no means necessary. If we take the origin at the level of the line of inflexion, we must write

$$y = b + \frac{\lambda k}{b} x_1 - B_3 x_1^3 + B_4 x_1^4 - B_5 x_1^5 + \dots ;$$

and we have

$$B_3 = \frac{\lambda^2 k}{6b^3}\left(1 + \frac{\lambda^2 k^2}{b^2}\right),$$

$$B_4 = \frac{B_3}{4k}\left\{1 + \lambda(2+\lambda)\frac{k^2}{b^2}\right\},$$

$$B_5 = \frac{B_3}{20k^2}\left\{1 + \lambda(1+2\lambda)\frac{k^2}{b^2} + \lambda^2(6 - 5\lambda + \lambda^2)\frac{k^4}{b^4}\right\}.$$

When $\lambda = 1$, we have

$$B_3 = \frac{k}{6b^3}\left(1 + \frac{k^2}{b^2}\right),$$

$$B_4 = \frac{B_3}{4k}\left(1 + 3\frac{k^2}{b^2}\right),$$

$$B_5 = \frac{B_3}{20k^2}\left(1 + 3\frac{k^2}{b^2} + 2\frac{k^4}{b^4}\right).$$

And lastly, when $k = b$, we have

$$B_3 = \frac{1}{3k^2}, \quad B_4 = \frac{1}{3k^3}, \quad B_5 = \frac{1}{10k^4},$$

or

$$y = k + x_1 - \frac{x_1^3}{3k^2} + \frac{x_1^4}{3k^3} - \frac{1x_1^5}{10k^4} + \dots .$$

If $x = -\frac{2}{5}k$, $\frac{dy}{dx}$ is very small arithmetically, and this value very nearly gives the level of the neck when $\lambda = 1$ and $k = b$. It is easy to see that, for any less negative value of x than that which makes $\frac{dy}{dx} = -\infty$, the series for $\frac{dy}{dx}$ and for y must both be convergent. The same follows *à fortiori* for similar positive values of x.

I do not see any obvious method of connecting the point at which $\frac{dy}{dx} = 0$ with that at which $\frac{d^2y}{dx^2} = 0$ for any *given* dome (no matter how given), except by getting out the figure of the dome graphically, or by a tentative process. The form of the differential equation, in which x does not appear explicitly, and which consequently is not altered by adding a constant to x, seems to me to indicate that this is a difficulty not likely to be surmounted except through the general solution of the equation.

It does not appear that the cone ever satisfies the conditions. But if the stress is considerable, the figure differs but little from a cone near the level of inflexion.

The introduction of a horizontal tension makes the problem wholly different from that of the arch of uniform stress, in which we can have $y=0$ and $\frac{dy}{dx}=\infty$ simultaneously. The form of this arch, or its invert, the catenary of equal tension, is well known. (See Venturoli's "Mechanics," Pratt's "Mechanical Philosophy," Todhunter's "Statics," and many others.) If we call t the thickness of the arch, and t_0 the thickness at the crown, its equation is

$$\frac{t}{t_0}=\epsilon^{\frac{x}{k}}=\sec\frac{y}{k}.$$

The curve is the same as that by which a small circle passing through the pole of a sphere is represented on Mercator's Chart.

Note on the Numerical Calculation of the Roots of Fluctuating Functions. By Lord Rayleigh, M.A., F.R.S.

[*Read June 11th*, 1874.]

There is an important class of functions, often occurring in physical investigations, whose numerical calculation is easy when the argument is either small or great. In the first case the function is readily calculated from an ascending series, which is always convergent and might be employed whatever the value of the variable may be, were it not for the length to which the calculations would run. When the argument is great, a series proceeding by descending powers is employed, whose character is quite different. In this case the series is of the kind called semi-convergent, though strictly speaking it is not convergent at all; for, when carried sufficiently far, the sum of the series may be made to exceed any assignable quantity. But, though ultimately divergent, it begins by converging, and when a certain point is reached the terms become very small. It can be proved that, if we stop here, the sum of the terms already obtained represents the required value of the functions, subject to an error which in general cannot exceed the last term included. Calculations founded on this series are therefore only approximate; and the degree of the approximation cannot be carried beyond a certain point. If more terms are included, the result is made worse instead of better. In the class of functions referred to, the descending series is abundantly adequate when the argument is large, but there will usually be a region—often the most interesting part of the whole—where neither series is very convenient. The object of the

present note is to point out how a part of the difficulty thence arising may sometimes be met.

Though the method in question is not limited to, it arose out of and is well exemplified by, the case of Bessel's functions, which are required among other purposes to represent the vibrations of a circular membrane, or of air contained within a cylindrical case. The roots of the equation obtained by equating the function to zero, give the possible periods of vibration, and the mechanics of the question show (as in all such cases) that no imaginary or complex root can occur.

If the function of the nth order be denoted as usual by $J_n(z)$, the ascending series is

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{z^2}{2.2n+2} + \frac{z^4}{2.4.2n+2.2n+4} - \dots \right\} \quad \dots\dots\dots\dots (1),$$

which is always ultimately convergent, whatever may be the values of z and n. When, however, z is considerable, the series becomes perfectly useless for arithmetical purposes. The descending series is

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left\{ 1 - \frac{(1^2-4n^2)(3^2-4n^2)}{1.2\,(8z)^2} + \dots \right\} \cos\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right)$$
$$+ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left\{ \frac{1^2-4n^2}{1.8z} - \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1.2.3\,(8z)^3} + \dots \right\} \sin\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right) \dots\dots\dots (2).$$

This series terminates when n is of the form (integer $+\frac{1}{2}$), and then of course constitutes the expression of the function in finite terms, but otherwise it runs on to infinity, and becomes ultimately, in all cases, divergent. In numerical calculations we are to include only the convergent part.

When z is very great, it is evident that the roots of the equation $J_n(z) = 0$, tend to the form

$$z = \frac{\pi}{4} + n\frac{\pi}{2} + \frac{2m-1}{2}\pi,$$

where m is an integer.

For purposes of explanation it will be sufficient to take the particular case of $n=0$, worked out by Prof. Stokes in his paper "On the numerical calculation of a class of definite integrals and infinite series."*

To calculate the roots, we find

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* Cambridge Transactions, vol. ix., 1850.

$$\frac{z}{\pi} = m - \tfrac{1}{4} + \frac{\cdot 050661}{4m-1} - \frac{\cdot 053041}{(4m-1)^3} + \frac{\cdot 262051}{(4m-1)^5} + \ldots\ldots\ldots\ldots (3),$$

which is adequate for all the roots, except the first, corresponding to $m=1$. If we denote the roots in order by p_1, p_2, &c., we get

$$p_2 \div \pi = 1{\cdot}757098, \qquad p_5 \div \pi = 4{\cdot}7527,$$
$$p_3 \div \pi = 2{\cdot}754568, \qquad p_6 \div \pi = 5{\cdot}7522,$$
$$p_4 \div \pi = 3{\cdot}7534, \qquad p_7 \div \pi = 6{\cdot}7519.$$

The higher the order of the root the more accurately it can be calculated from this series, but an accurate value of the first root can only be derived from the ascending series. To attain this end in the ordinary way requires a considerable expenditure of computation. My object is to show how the difficulty may be evaded.

Returning to the general function $J_n(z)$, and excluding the n zero roots, we see that

$$1 - \frac{z^2}{2\,.\,2n+2} + \frac{z^4}{2\,.\,4\,.\,2n+2\,.\,2n+4} - \ldots\ldots$$
$$= \left(1 - \frac{z^2}{p_1^2}\right)\left(1 - \frac{z^2}{p_2^2}\right)\left(1 - \frac{z^2}{p_3^2}\right)\ldots\ldots$$

must be an identical equation. Taking the logarithms of both sides, expanding, and equating like powers of z, we find that

$$\Sigma p^{-2} = \frac{1}{2^2\,.\,n+1}, \qquad \Sigma p^{-4} = \frac{1}{2^4(n+1)^2\,.\,n+2},$$

$$\Sigma p^{-6} = \frac{1}{2^5(n+1)^3\,.\,n+2\,.\,n+3}, \qquad \Sigma p^{-8} = \frac{5n+11}{2^8(n+1)^4(n+2)^2\,.\,n+3\,.\,n+4},$$

$$\Sigma p^{-10} = \frac{2\,.\,7n+19}{2^{10}(n+1)^5(n+2)^2\,.\,n+3\,.\,n+4\,.\,n+5},$$

the last result requiring a good deal of reduction.

If $n=0$, $\quad \Sigma p^{-2} = \frac{1}{4}, \quad \Sigma p^{-4} = \frac{1}{32}, \quad \Sigma p^{-6} = \frac{1}{192},$

$$\Sigma p^{-8} = \frac{11}{12288}, \quad \Sigma p^{-10} = \frac{19}{122880}.$$

If $n=1$, $\quad \Sigma p^{-2} = \frac{1}{8}, \quad \Sigma p^{-4} = \frac{1}{192}, \quad \Sigma p^{-6} = \frac{1}{3072},$

$$\Sigma p^{-8} = \frac{1}{46080}, \quad \Sigma p^{-10} = \frac{13}{8847360}.$$

Now Σp^{-10} must depend mainly on the first root, and being known numerically may be used to derive an accurate value of that root with the

assistance of approximate values of the other roots. We have, when $n=0$,

$$\begin{aligned}
\pi^{10} \div p_2^{10} &= \cdot00356484 \\
\pi^{10} \div p_3^{10} &= \cdot\quad 3976 \\
\pi^{10} \div p_4^{10} &= \cdot\quad 180 \\
\pi^{10} \div p_5^{10} &= \cdot\quad 17 \\
\pi^{10} \div p_6^{10} &= \cdot\quad 3 \\
\pi^{10} \div p_7^{10} &= \cdot\quad 1 \\
&\overline{\qquad\qquad} \\
&\cdot00360661
\end{aligned}$$

and

$$\frac{\pi^{10}}{p_1^{10}} = \frac{19\pi^{10}}{122880} - \Sigma_2^{\infty} \frac{\pi^{10}}{p^{10}},$$

so that $p_1^{-10} = \cdot0001546224 - \cdot000000038513 = \cdot0001545839.$

Seven significant figures in p_1^{-10} require only three in $\Sigma_2^{\infty} \frac{\pi^{10}}{p^{10}}$, so that all the roots after the third might be neglected, and the values of the second and third themselves are only required to be known roughly. The resulting value of p_1 is $p_1 = 2{\cdot}404826$.

The advantage of the present method would appear to depend upon the combined use of both the ascending and descending series.

The value of p_1 might of course be calculated from Σp^{-2}, or the sum of any other inverse powers. The advantage of using a high power is that the result becomes approximately independent of the other roots, and the tenth power was chosen as facilitating the numerical calculations. If a higher power than the tenth were required the algebraical reduction would become long, but this might easily be compensated by an advantage in the arithmetic.

As far as the 20th power inclusive, the expansion may be made without much difficulty by the aid of the table given on p. 775 of De Morgan's Differential and Integral Calculus. In this way the value of p_1 (for $n=0$) might be obtained to 6 or 7 significant figures without *any* allowance for p_2, p_3, &c.

The method is equally successful in its application to the case of $n=1$. We find $p_1^{-10} + \cdot000000003553 = \frac{13}{8847360} = \cdot000001469365,$

whence $$p_1 = 3{\cdot}831706 = 1{\cdot}219670\pi.$$

In cases where there is a difficulty about the *two* first roots, it might be possible to obtain the desired results by using the values of two of the sums of powers, *e. g.* Σp^{-10} and Σp^{-2}. In calculating the value of $\Sigma_3^{\infty} p^{-2}$, a large number of roots would have to be included, but the calculation could generally be facilitated by the use of approximate

formulæ. For example, in the case of $J_0(z)$, the roots after a certain point, say p_1, might be adequately represented by $z = m\pi$. Then

$$\Sigma_1^\infty p^{-2} = \Sigma_1^r p^{-2} - \Sigma_1^r \frac{1}{m^2\pi^2} + \Sigma_1^\infty \frac{1}{m^2\pi^2},$$

where the last term is $\frac{1}{6}$, by a known formula, which is in fact derivable from our general expression

$$\Sigma p^{-2} = \frac{1}{2^2 . n+1},$$

inasmuch as $$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin z.$$

A more important question arises as to whether this method can be applied to calculate the argument corresponding to a prescribed value (ϕ) of the function, other than zero. It might appear that this could always be done, inasmuch as when $J_0(z) = \phi$, another function ϕ, viz. $J_0(z) - \phi$, vanishes. But then the roots of this last function are not all real, and therefore cannot be calculated by the same method as before. Nevertheless, if ϕ were small, so that the first three or four roots were real, it would seem that the complex roots would be large enough to be dispensed with, as not sensibly influencing the value of Σp^{-10}, and then the first root could be deduced with sufficient precision from the values of the other real roots as calculated by aid of the descending series. I am not sufficiently acquainted with numerical calculations to say whether this application could ever be practically useful. If too laborious for the systematic tabulation of a function, it might perhaps be occasionally available as a control.

Nov. 22nd.

Prof. Cayley, to whom the preceding note was referred, has pointed out that a similar result may be attained by a method given in a paper by Encke, "Allgemeine Auflösung der numerischen Gleichungen," Crelle, t. xxii. (1841), pp. 193—248.

"Taking the equation

$$0 = 1 - ax + bx^2 - cx^3 + dx^4 - ex^5 + fx^6 - gx^7 + hx^8 - \ldots;$$

if the equation whose roots are the squares of these is

$$0 = 1 - a_1 x + b_1 x^2 - c_1 x^3 + \ldots,$$

then $$a_1 = a^2 - 2b, \quad b_1 = b^2 - 2ac + 2d, \quad c_1^2 = c^2 - 2bd + 2ae - 2f,$$
$$d_1^2 = d^2 - 2ce + 2bf - 2ag + 2h, \text{ \&c.};$$

and we may in the same way derive a_2, b_2, c_2, &c. from a_1, b_1, c_1, &c., and so on.

As regards the function

$$J_n(z) = \frac{z^n}{2^n . \Gamma(n+1)} \left\{ 1 - \frac{z^2}{2 . 2n+2} + \frac{z^4}{2 . 4 . 2n+2 . 2n+4} - \ldots \right\},$$

we have as follows:

$$a^{-1} = 2^2.n+1, \qquad a_1^{-1} = 2^4.(n+1)^2.n+2,$$

$$b^{-1} = 2^5.n+1.n+2, \qquad b_1^{-1} = 2^9.(n+1.n+2)^2.n+3.n+4,$$

$$c^{-1} = 2^7.3.n+1\ldots n+3, \qquad c_1^{-1} = 2^{13}.3.(n+1\ldots n+3)^2.n+4\ldots n+6,$$

$$d^{-1} = 2^{11}.3.n+1\ldots n+4, \qquad d_1^{-1} = 2^{19}.3.(n+1\ldots n+4)^2.n+5\ldots n+8,$$

$$e^{-1} = 2^{13}.3.5.n+1\ldots n+5,$$

$$f^{-1} = 2^{16}.3^2.5.n+1\ldots n+6,$$

$$g^{-1} = 2^{18}.3^2.5.7.n+1\ldots n+7,$$

$$h^{-1} = 2^{23}.3^2.5.7.n+1\ldots n+8,$$

$$a_2 = \frac{5n+11}{2^8.(n+1)^4(n+2)^2\,n+3\,.\,n+4},$$

$$b_2 = \frac{25n^2+231n+542}{2^{17}.(n+1\,.\,n+2)^4(n+3\,.\,n+4)^2\,n+5\ldots n+8},$$

$$*a_3 = \frac{429n^5+7640n^4+53752n^3+185430n^2+311387n+202738}{2^{16}\,(n+1)^8(n+2)^4(n+3.n+4)^2\,n+5\,.\,n+6\,.\,n+7\,.\,n+8}.$$

If $n=0$, $\quad \Sigma p^{-16} = a_3 = \dfrac{101369}{2^{27}.3^3.5.7} = p_1^{-16}$, suppose;

whence $\qquad p_1 = 2{\cdot}404825.$"

It will be seen that the expression a_3 for Σp^{-16} gives a very accurate value of p_1, without any allowance for the other roots.

If $n=1$, we have $\quad \Sigma p^{-16} = a^{-16} = a_3 = \dfrac{761376}{2^{30}.3^7.10^2.7}$;

whence $\qquad a = 3{\cdot}831690,$

and this is the approximate value of p_1. To get a corrected value, we may take $\qquad p_1 = a\,(1 + \frac{1}{16}a^{16}\Sigma_2^{\infty}p^{-16})$;

or, if all but p_2 may be omitted,

$$p_1 = a\left\{1+\tfrac{1}{16}\left(\frac{a}{p_2}\right)^{16}\right\}.$$

In the present case $p_2 = 7{\cdot}015$, so that

$$p_1 = a \times 1{\cdot}0000039 = 3{\cdot}831705.$$

It may be observed that the value obtained without correction for the higher roots is always an under estimate.

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* I have altered the coefficient of n^3 in the numerator from 53572 to 53752.

Spherical and Cylindric Motion in Viscous Fluid.

By J. H. Röhrs.

[*Read May 14th*, 1874.]

Rotating Sphere filled with Viscous Fluid.

It is more than three years ago since my attention was attracted to the problem of a rotating sphere filled with viscous fluid. I attacked it with the idea of applying the results of the investigation to determine roughly what changes would occur in the general phenomena of Precession and Nutation if the earth's interior were composed of viscous fluid instead of being, as Sir W. Thomson and other competent mathematicians assure us, rigidly solid. I was not aware at that time that Professor Helmholtz had occupied himself with the same problem for the purpose of calculating the value of μ', the coefficient of fluid friction, from experiments on a rotating sphere filled with fluid. Since then, and at the request of the referee, I have endeavoured to read Helmholtz's paper; but my knowledge of German was not sufficient for the task, and though I could make out tolerably well the analysis, which seemed elegant and neat, I could but half comprehend the nature of the experiments and the machinery employed. From roughly carrying the approximation to the second order of small quantities, I was led to conclude, however, that Helmholtz's method, as far as I could understand his data, might give, if carefully conducted, sufficiently exact results, though not without sensible error perhaps; and I should suppose that Professor Helmholtz has taken the necessary precautions, but I cannot venture an opinion on that point. However, that question does not concern me. I have been trying very hard to get some general method of estimating the amount of error in the results given by the approximate equations in the case of the earth. Professor Stokes advised me to try "error forces"; and Professor Maxwell sent me an elegant little investigation, based on a similar method, and having for its object to arrive at a limit of the error committed by neglecting terms involving the squares of the velocities—*i. e.*, by neglecting the terms that differentiate $\frac{du}{dt}$ from $\frac{\Delta u}{\Delta t}$, leaving out, in fact, $u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$, &c.

But I cannot myself see any other way than by going to a second approximation, to test the accuracy of the first. If the neglected terms turn out small in comparison with those retained, well and good; otherwise the equations we started with are useless, and do not apply

to this case. If the first approximation be good, a second can be arrived at, if necessary, without much difficulty; but if it be not good, I do not think we shall be able to make much use of it by the introduction of fictitious forces. I have added two or three pages on the discussion of $\frac{d\omega}{dt} = \mu'\left(\frac{d^2\omega}{dr^2} + \frac{3}{r}\frac{d\omega}{dr}\right)$,

where $r^2 = x^2 + y^2$, and the axis of z is the axis of rotation.

This is the equation to cylindrical motion, or the motion of fluid in a closed cylinder, the periphery being rough, but the plane surfaces that close the top and bottom perfectly smooth. This equation is true without restriction as to the values of ω and $\frac{d\omega}{dr}$, and explains nicely the behaviour of eddies in ponds, tea-cups, &c., and such-like cases of whirling motion. When the radius of the cylinder is supposed infinite, the solution can be put under the form of a definite integral, an initial disturbance being supposed impressed on the fluid, which is left to itself after that, and the motion not kept up by any external forces.

On the Rotation of a Hollow Sphere filled with Viscous Fluid, and made to Rotate about its Centre under the action of external impressed given Periodic Forces.

A hollow sphere, the mass of the solid shell of which is supposed inconsiderable in comparison with that of the fluid interior, is filled with viscous incompressible fluid having the same density throughout, and made to slowly rotate about its centre, supposed fixed, by external impressed periodic forces applied to the shell: to determine the motion.

We shall assume that no part of the motion of the fluid is radial, or in direction of the radius; and *prove that if* terms such as $u\frac{du}{dx}$, &c., or terms depending on the squares of the velocity, may be neglected, that the equations of fluid motion may be satisfied by values of u, v, w; the velocities parallel to the axes of x, y, z respectively by values of

$u, v, w,$

$$u = \omega_2 z - \omega_3 y,$$
$$v = \omega_3 x - \omega_1 y,$$
$$w = \omega_1 y - \omega_2 x,$$

where ω_1, ω_2, ω_3 are functions of r, the radius, only, and *not* of θ and ϕ, the angles r makes with the axis of z, and that the plane through r and axis of z make with plane (zx).

x, y, z, coordinates of any point in the fluid at distance r from the centre of the sphere; a = radius of sphere; ρ the density, which will be generally taken as 1; and μ the coefficient of friction; and $\frac{\mu}{\rho} = \mu'$.

It must be observed that the equations we shall arrive at are only true approximately, and that, if the variations of w exceed a certain limit, so that the neglected terms are not small in comparison with those retained, the method fails. For the purposes to which I applied these equations, a rough calculation of what might be about the least value of μ' in the earth's interior that would not perceptibly affect the constants of Precession and Nutation, the equations are sufficiently true, as a *very* small variation in $\frac{d\omega_1}{dr}$, $\frac{d\omega_2}{dr}$, $\frac{d\omega_3}{dr}$ would be perceptible; and the conclusion to be arrived at, that μ' is enormous, and beyond any comparison with the value in any molten metal or fluid, is no doubt true; but if μ' had been comparable with its value in water, the equations would not in the least represent the state of the earth's interior.

If w_1, w_2, w_3 be functions of r only, where $r^2 = x^2+y^2+z^2$, and not of θ and ϕ, the ordinary semi-polar angles, we observe that the equation of continuity is satisfied; for

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

is satisfied, since

$$z\frac{d\omega_2}{dx}-y\frac{d\omega_3}{dx}+x\frac{d\omega_3}{dy}-z\frac{d\omega_1}{dy}+y\frac{d\omega_1}{dz}-x\frac{d\omega_2}{dz}=0 \quad \ldots\ldots (2);$$

since, if ω_1, ω_2, ω_3 be functions of r only, $\frac{d\omega_1}{dy}=\frac{d\omega_1}{dr}\frac{y}{r}$ &c. = &c., and equation (2) is satisfied.

The equations

$$\left.\begin{aligned}\frac{du}{dt}+\frac{dp}{dx}&=\mu'\Delta^2 u\\ \frac{dv}{dt}+\frac{dp}{dy}&=\mu'\Delta^2 v\\ \frac{dw}{dt}+\frac{dp}{dz}&=\mu'\Delta^2 w\end{aligned}\right\},$$

where ρ is taken $=1$ for shortness, and $\Delta^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$,

give, by elimination of p,

$$\Delta^2\left\{\frac{du}{dt}-\mu'\Delta^2 u\right\}=0, \quad \&c.\ \&c.;$$

and any values of u, &c. that satisfy $\frac{du}{dt}-\mu'\Delta^2 u=0$, will clearly satisfy

$$\Delta^2\left\{\frac{du}{dt}-\mu'\Delta^2 u\right\}=0, \quad \&c.\ \&c.$$

Now $u=\omega_2 z-\omega_3 y$; and observing that $\frac{d\omega_3}{dy}=\frac{y}{r}\frac{d\omega_3}{dr}$, &c.,

$$\rho\left\{\frac{d}{dt}(\omega_2 z-\omega_3 y)\right\}=\mu\left\{z\left(\frac{d^2\omega_2}{dr^2}+\frac{4}{r}\frac{d\omega_2}{dr}\right)-y\left(\frac{d^2\omega_3}{dy^2}+\frac{4}{r}\frac{d\omega_3}{dy}\right)\right\};$$

and since $z=r\cos\theta$ and $y=r\sin\theta\cos\phi$, and w_3 and w_2 are functions of r only, and that

$$z\left\{\frac{d\omega_2}{dt}-\frac{\mu}{\rho}\left(\frac{d^2\omega_2}{dr^2}+\frac{4}{r}\frac{d\omega_2}{dr}\right)\right\}=y\left\{\frac{d\omega_3}{dt}-\frac{\mu}{\rho}\left(\frac{d^2\omega_3}{dr^2}+\frac{4}{r}\frac{d\omega_3}{dr}\right)\right\},$$

$$\frac{d\omega_2}{dt}-\frac{\mu}{\rho}\left(\frac{d^2\omega}{dr^2}+\frac{4}{r}\frac{d\omega_2}{dr}\right)=0;$$

and similar equations for w_3 and w_1, derived from this and the other equations, will be obtained satisfying $\frac{du}{dt}-\mu'\Delta^2 u=0$ and the other two like equations, and therefore satisfying the general equations containing $\frac{dp}{dx}$, and from which these were derived.

Hence the differential equations for the velocities, and also the equation of continuity, being satisfied by ω_1, ω_2, ω_3, they are possible solutions when the motion is so slow that the *variations* of ω_1, ω_2, ω_3 are inconsiderable in comparison with ω_1, ω_2, ω_3. In this case, for a sphere started from rest about its centre, and in which no initial radial motion has been impressed, ω_1, ω_2, ω_3 will satisfy the conditions of the problem, and the fluid will roll over itself in concentric shells having the same angular velocities about the three fixed axes at all points at the same distance from the centre.

If now w_3, to take one of the angular velocities, be generated by a periodic force such that the value of ω_3 would be $n\sin pt$ if the contents of the globe had been solid instead of being fluid, ω_3 will be represented by a function, $\mathrm{P}\sin pt+\mathrm{Q}\cos pt+\mathrm{V}$, where V will be of the form $\Sigma.\mathrm{C}_m e^{-mt}$ (m and C_m presently to be determined), and C_m, a $f(r)$.

If we are considering the motion a long time after the globe has been started from rest, V may be neglected, as it tends to zero as t tends to infinity. Then from the fundamental equation for $\frac{d\omega_3}{dt}$, since

$$\omega_3=\mathrm{P}\sin pt+\mathrm{Q}\cos pt+\mathrm{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$p\mathrm{P}=\mu'\left(\frac{d^2\mathrm{Q}}{dr^2}+\frac{4}{r}\frac{d\mathrm{Q}}{dr}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

$$-p\mathrm{Q}=\mu'\left(\frac{d^2p}{dr^2}+\frac{4}{r}\frac{dp}{dr}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

These equations may be written

$$\mu'\left\{\frac{d^2}{dr}(\mathrm{Q}r^2)-2\mathrm{Q}\right\}=\mathrm{P}pr^2,$$

$$\mu'\left\{\frac{d^2}{dr}(\mathrm{P}r^2)-2\mathrm{P}\right\}=-\mathrm{Q}pr^2.$$

Let $Pr^2 = U'$, $Qr^2 = V'$, and let $\frac{p}{\mu'} = 2m'^2$.

If now $U' = e^{-\lambda' r}\left(1+\frac{1}{\lambda' r}\right)$,

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) U' = \lambda'^2 U';$$

and eliminating V′ between the two equations (7) and (8),

$$-4m'^4 U' = \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 U'.$$

If $\lambda'^4 = -4m'^4$, this is satisfied; therefore

$$U' = ce^{-\lambda' r}\left(1+\frac{1}{\lambda' r}\right), \text{ where } \lambda' = m'(\pm 1 \pm \sqrt{-1}).$$

Changing impossibles into sines and cosines, this gives

$$\begin{aligned} U' = \quad & e^{m'r}\left\{c_1 \cos m'r\left(1-\frac{1}{2m'r}\right) + c_2 \frac{\cos m'r}{2m'r}\right\} \\ & + e^{m'r}\left\{c_2 \sin m'r\left(1-\frac{1}{2m'r}\right) - c_1 \frac{\sin m'r}{2m'r}\right\} \\ & + e^{-m'r}\left\{d_1 \cos m'r\left(1+\frac{1}{2m'r}\right) + d_2 \frac{\cos m'r}{2m'r}\right\} \\ & + e^{-m'r}\left\{d_2 \sin m'r\left(1+\frac{1}{2m'r}\right) - d_1 \frac{\sin m'r}{2m'r}\right\}. \end{aligned}$$

And V′ will have the same form; only if c_1', c_2', d_1', d_2' be its corresponding constants, $c_1' = -c_2$, $c_2' = c_1$, $d_1' = -d_2$, $d_2' = d_1$.

The condition that $\frac{dP}{dr}$ or $\frac{d}{dr}\frac{U'}{r^2}$ may $= 0$ when $r = 0$ gives two equations to determine c_1, c_2, d_1, d_2, and two more are given from the conditions at the shell.

These conditions are furnished by the necessity that the total integral $\iiint r^4 \frac{d\omega_3}{dt} \sin^3\theta \, d\phi \, d\theta \, dr$, taken throughout the globe, must be the same as if the globe were solid; or,

$$\tfrac{8}{3}\pi\mu' a^4 \frac{d\omega_3}{dr}_{r=a} = \tfrac{8}{15}\pi a^5 np \cos pt \quad \ldots\ldots\ldots\ldots\ldots (10).$$

Hence the two conditions:

(1) The part of $\frac{d\omega_3}{dr}_{r=a}$ which is a factor of $\sin pt$ must be put $= 0$ and (2) the part which is a factor of $\cos pt$ equated to $\frac{8}{15}\pi a^5 np$.

Equation (10) is obtained by remembering that

$$\frac{d\omega_3}{dt} = \mu'\left(\frac{d^2\omega_3}{dr^2} + \frac{4}{r}\frac{d\omega_3}{dr}\right),$$

and $$r^4\frac{d^2\omega_3}{dr^2}+4r^3\frac{d\omega_3}{dr}=\frac{d}{dr}\left(r^4\frac{d\omega_3}{dr}\right).$$

U is now determined; but we have to determine V, calling P $\sin pt$ + Q $\cos pt$, U.

Let U+V be the complete value of ω_3.

Now the conditions at the limits $r=0$ and $r=a$ at surface, imposed on ω_3, are both satisfied by U; therefore $\frac{dV}{dr}=0$ both when $r=0$ and when $r=a$.

Let $V_m e^{-qt}$ be a particular value of V, where V_m is a $f(r)$ only.

Then $$\frac{d^2V_m}{dr^2}+\frac{4}{r}\frac{dV_m}{dr}+\frac{q}{\mu'}V_m=0.$$

Let $\frac{q}{\mu'}=m^2$, $V_m r^2=z$; therefore

$$\frac{d^2z}{dr^2}+m^2z-\frac{2z}{r^2}=0;$$

and the solution of this is

$$z=A\cos(mr)-\frac{A\sin(mr)}{mr}+B\sin(mr)+\frac{B\cos(mr)}{mr}.$$

The condition that $\frac{dV_m}{dr}=0$ when $r=0$ makes B=0; and the condition that $\frac{dV_m}{dr}=0$ when $r=a$ gives $\tan(ma)=\frac{3ma}{3-m^2a^2}$.

This has an infinite number of real positive roots, and the solution for V is $V=\Sigma(C_mV_m e^{-qt})$, $(q=\mu'm^2)$.

To determine V_m, we observe that, it being a form of Poisson's V (see Gregory's Examples, p. 521), and also because $\frac{dV_m}{dr}=0$ at the limits $r=0$ and $r=a$, between those limits any continuous function $f(r)$ can be readily developed in a series $\Sigma.C_mV_m$.

For $\int_0^a V_mV_n r^4dr=0$ always, unless $m=n$.

Hence, if $f(r)$ be put $=\Sigma.C_mV_m$, multiplying both sides by r^4V_m, and integrating, $$\int_0^a r^4V_m f(r)=C_m\int_0^a V_m^2 r^4,$$

all the other terms going out, whence C_m.

Suppose, then, at time $t=0$, ω_3 be given as a known $f(r)$; therefore

$$f(r)=U_{t=0}+V_{t=0}.$$

$U_{t=0}$ is known, therefore $f(r)-U_{t=0}$ is known, and can be expanded in a series $\Sigma.D_mV_m$, and V in a series $\Sigma.B_mV_m e^{-mt}=\Sigma.B_mV_m$ $(t=0)$; therefore $\Sigma.D_mV_m=\Sigma.B_mV_m$; therefore $B_m=D_m$, and is known.

It may be observed that, because $\tan\theta = -\tan(-\theta)$, for every root m there is a root $-m$ in the equation $\tan ma = \frac{3ma}{3-m^2a^2}$.

But V_m and V_{-m} have the same value,

$$\frac{A}{r^2}\left\{\cos(\pm mr) - \frac{\sin(\pm mr)}{\pm mr}\cdot\frac{1}{r}\right\};$$

so that we get only one value V_m for $\pm m$.

Also the series for V can be readily shown to be convergent, being asymptotic with a Fourier's Series for very great values of m, and coinciding with it when m is infinite.

If the external force be not a periodic function of t the time, it may be expanded between limits $t=0$ and $t=\tau$ in a Fourier's Series of sines and cosines, and the solution then will be of the same form as above.

It may be observed that Helmholtz's equation is a particular solution of

$$\Delta^2\left\{\frac{du}{dt} - \mu'\Delta^2(u)\right\} = 0.$$

If $u = \psi_1 + \psi_2$ be the complete solution,

$$\psi_1 \text{ satisfies the equation } \frac{d\psi_1}{dt} - \mu'\Delta^2\psi_1 = 0,$$

$$\psi_2 \quad \text{,,} \quad \text{,,} \quad \mu'\Delta^2\psi_2 = 0,$$

as Professor Stokes has shown. (See Paper on the Effect of the Internal Friction of Fluids on the Motion of Pendulums, by G. G. Stokes, Transactions of the Cambridge Philosophical Society, 1851.)

In Helmholtz's equation, $dp=0$ or $p=c$, (where under p may be included V, the potential of natural forces, such as the force of gravity; so that, if p' be the fluid pressure, $p = p' + V$). Helmholtz's equation is therefore equivalent to putting $\psi_2 = 0$; and as the conditions at surface are satisfied thereby, this is legitimate.

The application of these results to Precession and Nutation is obvious.

Neglect, for the moment, the small terms of Nutation, and refer the earth at any instant to three axes fixed in direction in space.

Let one axis, the axis of z, be perpendicular to the plane of the ecliptic, and let a plane through this axis and the earth's polar axis of rotation, or instantaneous axes, make an angle pt with the axis taken for the axis of x, at time t. Then, because the earth's pole rotates about the pole of the ecliptic in about 25,000 years,

$p \times 25{,}000$ years $= 2\pi$ (units of measurement, seconds and feet);

and the angular velocity at time t about the axes of x, y, z will be (if n be the earth's angular velocity about her polar axis, and i the inclination of the axis of ecliptic to that of equator) $n\sin i\cos pt$, $n\sin i\sin pt$, $n\cos i$ respectively. Hence, if we consider the motion of a spherical core of the earth, concentric with the earth's centre, touching the poles and leaving out the protuberance due to the slightly spheroidal figure

of the earth, the angular motion of this spherical core about the axis of x will be maintained by a tangential force acting on it through its connection with the thin outer rind, and communicating to it an angular velocity $n \sin i \sin pt$ about the axis of y.

Suppose now this core, instead of being solid, consisted of viscous fluid, and that the same tangential external force were to act on it; and compare the results, as indicated by the previous investigation, with what is actually the case, *i. e.* when the core is solid, and the difference will be great indeed; unless am is nearly zero, *i. e.* unless $a\sqrt{\frac{p}{2\mu'}}$ is nearly zero, where a is the earth's radius to the pole, and the radius therefore of the imaginary core.

Put $$\mu' = \frac{pa^2}{2b^2}, \quad \text{therefore} \quad m = \frac{b}{a}$$

(b is a constant introduced to render the arithmetic more easy and striking).

Suppose now, to take an instance, $b = 10$; therefore

$$\mu' = \frac{pa^2}{200} = \frac{p\,(5280 \times 4000)^2}{200} \quad (a = 4000 \text{ miles});$$

and, taking the period of $\sin pt$ as 25,000 years,

$$\mu' = \frac{2\pi}{25{,}000 \times 31{,}500{,}000} \times \frac{(5280 \times 4000)^2}{200} = 18 \text{ nearly in feet and seconds.}$$

(31,500,000 is the number of seconds in one year, nearly.) In this example the shell is supposed to stick to the fluid core, and not to slip over it.

Now the value of μ' in feet and seconds, given by Professor Stokes for water, is ·000022 nearly.

If $\mu' = 18$, the fluid must be far more viscous than any molten substance we are acquainted with; yet, without absolutely calculating out, by a tedious arithmetical process, the result, it is sufficient to substitute $\frac{10}{a}$ for m in the equations for U′ and V′ to see how altogether different is the motion of the globe filled with viscous fluid from what it would have been if the globe had been solid throughout.

U′ and V′ rapidly decrease from the surface towards the centre, and therefore at the surface are much greater than they would be if the globe were solid.

For such a small value of μ' as 18, it is true that the equation gives widely erroneous results; but it is available to prove the discrepancy, though not exact enough to measure that discrepancy with accurateness. Common sense is sufficient to show that the earth's core cannot be fluid, and that one effect of fluidity would be to diminish enormously the value of Mk^2 in the terms of Precession and Nutation; and yet this

idea of a fluid molten interior enveloped by a thin shell is still constantly reproduced in educational works.

When μ' is very great, so that $\frac{\mu'}{a}$ is $> p$, by simply expanding in ascending powers of $\frac{r}{a}$, ω, when ω is put $= \mathrm{P} \sin pt + \mathrm{Q} \cos pt$, and equating coefficients, a converging series is readily obtained for ω, and is handy for calculation.

In this way, if $\quad \mathrm{P} = n' + \mathrm{A}_2 r^2 + \mathrm{A}_4 r^4 + \&c.,$

$$\mathrm{Q} = m' + \mathrm{B}_2 r^2 + \mathrm{B}_4 r^4 + \&c.,$$

and $\frac{p}{\mu'}$ be put $= \lambda$, and n be the value of ω at the shell, supposing the core had been solid, and λa^2 be < 1 (a, radius of shell),

$$n' = n, \quad m' = -\lambda \frac{a^2 n}{14} \text{ nearly};$$

and the value of ω at the shell will now be

$$\omega = n \sin pt - \lambda \frac{a^2}{14} n \cos pt \quad \text{nearly.}$$

If $\lambda a^2 = 1$, μ' will be about 3300 in feet and seconds, and I should think that μ' would not be so great as that in any viscous mass, except perhaps in half-cooled and solidified lava, in which it may be much more; certainly not in any fairly *fluid* mass. But a much greater value than this of μ' would probably be detected by astronomical observation.

If μ' be comparable with its value in water, I think that at less than 100 miles or so from the crust of the earth the motion of the fluid would be all but unaffected by the motion of the crust.

However, without entering into the calculation, it is impossible to guess the extent to which the movement of the shell might affect the fluid within. I may perhaps try and perform this calculation; but it will be a repulsive and an useless task, for the general results can be clearly foreseen.

Cylindric Motion.

With a very short discussion of the properties of the kindred equation,

$$\frac{d\omega}{dt} = \mu' \left(\frac{d^2\omega}{dr^2} + \frac{3}{r} \frac{d\omega}{dr} \right) \quad (\text{where } r^2 = x^2 + y^2),$$

which is the equation to cylindrical motion, I will conclude.

This form of motion is *possible* in a cylinder of which the periphery is rough, and the top and bottom covers that close it perfectly smooth planes perpendicular to the axis; or we may suppose the cylinder to be of infinite length, and extended to an infinite distance on both sides of the plane of xy; the axis of z being the axis of the cylinder.

We shall then have

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+\frac{1}{\rho}\frac{dp}{dx}=\mu'\Delta^2(u),$$
$$\frac{dv}{dt}+v\frac{dv}{dy}+u\frac{dv}{dx}+\frac{1}{\rho}\frac{dp}{dy}=\mu'\{\Delta^2(v)\}.$$

If natural forces of the form $\frac{dV}{dx}$, &c. be supposed to act on the fluid, they may be included in the differentials $\frac{dp}{dx}$, $\frac{dp}{dy}$; and p', the fluid pressure, will then be given by

$$\frac{1}{\rho}\frac{dp}{dx}=\frac{1}{\rho}\frac{dp'}{dx}+\frac{dV}{dx},\ \&c.=\&c.\ \&c.$$

Now if $u=-\omega y$, $v=\omega x$ (ω a function of r only), $\frac{du}{dx}+\frac{dv}{dy}=0$, as it ought; and putting $\rho=1$,

$$\frac{du}{dt}+\frac{dp}{dx}-\omega^2x=\mu'\Delta^2(u),$$
$$\frac{dv}{dt}+\frac{dp}{dy}-\omega^2y=\mu'\Delta^2(v).$$

Hence, as before, putting for u its value $-\omega y$, and for v its value ωx, these equations are satisfied by

$$\frac{d\omega}{dt}=\mu'\left(\frac{d^2\omega}{dr^2}+\frac{3}{r}\frac{d\omega}{dr}\right);$$

and then

$$\frac{dp}{dx}-\omega^2x=0,$$
$$\frac{dp}{dy}-\omega^2y=0;$$

and hence

$$\frac{d}{dy}\left(\frac{dp}{dx}\right)=\frac{d}{dx}\left(\frac{dp}{dy}\right),\quad\text{as it ought,}$$

and

$$\frac{d}{dz}\left(\frac{dp}{dx}\right)=\frac{d}{dz}\left(\frac{dp}{dy}\right)=0,\quad\text{as they ought.}$$

Consequently, if the equation for ω satisfies, as it does, the conditions at the limiting surfaces, it is a true equation, and true whatever be the magnitude of the motion.

We should have obtained the same equations and conditions if ω had been assumed a $f(\sqrt{x^2+y^2+z^2})$; only then, of course, $\frac{3}{r}\frac{d\omega}{dr}$ would have been replaced by $\frac{4}{r}\frac{d\omega}{dr}$, as we have found it; also $\frac{d}{dy}\frac{dp}{dx}$ would have been $=\frac{d}{dx}\frac{dp}{dy}$, but the condition $\frac{d}{dz}\frac{dp}{dx}=0$ would not have been

fulfilled, and the error $\omega \frac{d\omega}{dz}$ is the measure of the error in Helmholtz's equation, which is only applicable when the errors arising from this term may be neglected.

If the radius of the cylinder be $= a$, putting $\omega = \phi e^{-\alpha t}$, we obtain for ω a particular value given by the equation

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} = -\frac{a}{\mu'}\phi = -n\phi \text{ suppose.}$$

This equation cannot be integrated in finite terms. To obtain *accurate* results with a minimum of labour, we should have to treat it by Professor Stokes' method, and develope ϕ both in ascending and descending series of powers of r; the constants in the two series being connected by his process—which, though the best that could be devised, leads to severe work.

When the radius of the cylinder is infinite, the solution can be exhibited in the form of a definite integral—more pretty to look at, than useful to work with.

I will indicate the steps by which it may be obtained; and, as they are nearly the same as those by which the solution of $\frac{d^2y}{dt^2} = y\frac{d}{dx} \cdot x\frac{dy}{dx}$ is obtained,—a problem set last year in the Smith's Prize Papers by Professor Stokes, and the solution of which he was kind enough to send me,—I will only point out where the treatment varies a little from that of the "dangling rope" problem, of which $\frac{d^2y}{dt^2} = y\frac{d}{dx} \cdot x\frac{dy}{dx}$ is the equation.

Put $$\phi = A_0\{1 + A_2r^2 + A_4r^4 + \&c.\}.$$

Then it will be found, by equating coefficients, that

$$\phi = A_0\left\{1 - \frac{nr^2}{2.4} + \frac{n^2r^4}{2.4.4.6} - \&c.\right\}$$

$$= A_0\left\{1 - \frac{n}{2}\frac{r^2}{2^2} + \frac{n^2}{3}\frac{r^4}{2^2.4^2} - \&c.\right\}$$

(ϕ must $= 0$ when $r = \infty$, and $\frac{d\phi}{dr} = 0$ when $r = 0$); therefore

$$\frac{n\phi}{A_0} = \left\{n - \frac{n^2}{2}\frac{r^2}{2^2} + \frac{n^3}{3}\frac{r^4}{2^2.4^2} - \&c.\right\},$$

therefore $$\frac{d}{dn}\frac{n\phi}{A_0} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos\left(\sqrt{n}\, r\cos\theta\right) d\theta;$$

therefore $$\phi = \frac{2}{\pi}\frac{A_0}{n}\int_0^n\int_0^{\frac{1}{2}\pi} \cos\left(\sqrt{n}\, r\cos\theta\right) d\theta\, dn.$$

If $f(r)$ be the value of ω when $t=0$, and ω be the complete value of w, writing $\psi(n)$ for A_0, and $\varpi(n)$ for $1 - \frac{\nu r^2}{2 \cdot 2^2}$ + &c.,

$$\omega = \int_0^\infty \psi(n)\, \varpi(n)\, dn \,.\, e^{-\mu' n t},$$

and

$$f(r) = \int_0^\infty \psi(n)\, \varpi(n)\, dn.$$

It is only in $\frac{1}{n} \int_0^n \int_0^{\frac{1}{2}\pi} \cos(\sqrt{n}\, r \cos\theta)\, d\theta\, dn$

that there is a step different from those of the "rope problem." This integral will have to be evalued for infinite values of r, and requires a little treatment. Suppose r very great, but not infinite, and n also finite. Then $\int_0^n \int_0^{\frac{1}{2}\pi}$ is the same as $\int_0^{\frac{1}{2}\pi} \int_0^n$; also, if r be as great as we please,

$$\int_0^{\frac{1}{2}\pi} \cos(\sqrt{n}\, r \cos\theta)\, d\theta = \int_0^\beta \cos(\sqrt{n}\, r \cos\theta)\, d\theta,$$

where β is as small as we please, only r must be so large that $n^{\frac{1}{2}} r^{\frac{1}{2}} \beta$ shall be as great as we please,—this is easily proved, and I take it for granted that it is known,—and

$$\int_0^n \cos(\sqrt{n}\, r \cos\theta)\, dn = \frac{2 \sin(\sqrt{n}\, r \cos\theta)\, \sqrt{n}}{r \cos\theta} + \frac{2 \cos(\sqrt{n}\, r \cos\theta)}{r^2 \cos^2\theta}.$$

This is easily seen by putting $\sqrt{n} = z$, therefore $dn = 2z\, dz$, &c., and the second term may be neglected in comparison with the first, because r is *very* great; therefore

$$\frac{1}{n} \int_0^n \int_0^{\frac{1}{2}\pi} \cos(\sqrt{n}\, r \cos\theta)\, d\theta\, dn = \frac{2}{\sqrt{n}} \int_0^\beta \frac{\sin(\sqrt{n}\, r \cos\theta)}{r \cos\theta}\, d\theta = \mathrm{I}\ \text{(suppose)},$$

when r is *very* great.

Put now $n^{\frac{1}{2}} r^{\frac{1}{2}} \theta = q$, therefore $\theta = q r^{-\frac{1}{2}} n^{-\frac{1}{2}}$, $d\theta = dq r^{-\frac{1}{2}} n^{-\frac{1}{2}}$, and

$$\mathrm{I} = \frac{2}{\sqrt{n}} \int_0^{r^{\frac{1}{2}} n^{\frac{1}{2}} \beta} \frac{\sin(\sqrt{n}\, r) \cos\left(\frac{q}{r^{\frac{1}{2}} n^{\frac{1}{2}}}\right)}{r^{\frac{1}{2}} n^{\frac{1}{2}} \cos\left(\frac{q}{r^{\frac{1}{2}} n^{\frac{1}{2}}}\right)}\, dq.$$

Hence, because β can be made as small as we please, and ultimately vanish, expanding $\cos\left(\frac{q}{r^{\frac{1}{2}} n^{\frac{1}{2}}}\right)$, and observing that between the limits of integration $\frac{q}{r^{\frac{1}{2}} n^{\frac{1}{2}}}$ is indefinitely small,

$$\varpi(n) = \frac{2}{\pi} \mathrm{I} = \frac{4}{\pi n^{\frac{3}{2}} r^3} \left[\int_0^\infty \left\{ \sin(\sqrt{n}\, r) \cos\left(\frac{q^2}{2}\right) - \cos\sqrt{n}\, r \sin\left(\frac{q^2}{2}\right) \right\} dq \right]$$

$$= \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}} n^{\frac{3}{4}} r^{\frac{3}{2}}} \left\{ \sin\left(\sqrt{n}\, r - \frac{\pi}{4}\right) \right\} \text{ ultimately, in limit.}$$

From this point, proceeding exactly as in the rope problem, remembering that $-nr^3\varpi(n) = \frac{d}{dr}\left\{\frac{r^3 d(\varpi n)}{dr}\right\}$,

$A_0 = \psi(n)$ will be found $= \frac{n}{8} \int_0^\infty r'^3 f(r')\, \varpi(n')\, dr'$,

and $\omega = \int_0^\infty \cdot \left\{ \int_0^\infty r'^3 f(r')\, \varpi(n')\, dr' \right\} \frac{n}{8}\, \varpi(n)\, e^{-nt\mu'}\, dn.$

Putting $r^2 = z$, we obtain a series

$$\phi = A_0 e^{-\frac{1}{8}(nr^2)} \left\{ 1 - \frac{r^4 . n^2}{6 . 8^2} - \frac{r^6 . n^3}{18 . 8^3} - \frac{r^8 . n^4}{46 . 8^4} - \frac{13 r^{10} . n^5}{2700 . 8^5} \pm \&c. \right\}.$$

If $\frac{nr^2}{8}$ be not > 1, this is a rapidly converging series; and, for the illustration of eddies, such as we sometimes see on the surface of a pond, or even of the tea in a tea-cup, the above series, taking only one and two terms, is amply sufficient.

Numerous properties of such eddies might be proved, but I will only point out one or two.

First, if $\frac{nr^2}{8}$ be < 1 or a fraction, it is clear that ϕ increases toward the centre; this is constantly observed.

Secondly, taking only the first term, when $\frac{nr^2}{8}$ is very small,

$$\omega = \Sigma A_0 e^{-\frac{1}{8}(nr^2)} . e^{-\mu' nt} = \Sigma \left\{ A_0 e^{-\left(\frac{1}{8}r^2 n + \mu' t\right)} \right\}.$$

If ω_0 be the value of ω at time $t=0$,

$$\omega_0 = \Sigma \left\{ A_0 e^{-n\left(\frac{1}{8}r^2\right)} \right\} \text{ (suppose).}$$

So that, if $\frac{r'^2}{8} = \frac{r^2}{8} + \mu' t$, $\omega_0 = \omega$, or $r'^2 - r^2 = 8\mu' t$.

If now $r=0$, $t = \frac{r'^2}{8\mu'}$, or the time it takes to reduce the angular velocity at the centre at a given time, to what it is at that given time at a distance r' from the centre $= \frac{r'^2}{8\mu'}$. And many other properties might

be proved which, though interesting in themselves, are not of sufficient importance to discuss in this Journal.

Note.—In the method of obtaining the definite integral given above, as well as that of the "rope problem," we have to evaluate $\int_{-\infty}^{\infty} \frac{\sin r\theta\, d\theta}{\theta}$. I found difficulty in proving this equal to π, its proper value, to my satisfaction; but I think it may be shown to be true,—thus:

$$\int_0^\infty dx \,.\, x^{n-1} \sin rx = \frac{\Gamma(n)}{r^n} \sin\left(\frac{n\pi}{2}\right),$$ (See Gregory's Examples, p. 478,)

and $$\Gamma n \,.\, \Gamma(1-n) = \frac{\pi}{\sin n\pi};$$

therefore

$$\int_0^\infty dx \,.\, x^{n-1} \sin rx = \frac{\pi \,.\, \sin\left(\frac{n\pi}{2}\right)}{r^n\, \Gamma(1-n) \sin(n\pi)} = \frac{\pi}{2} \text{ in limit, when } n=0;$$

therefore $$\int_{-\infty}^{\infty} dx \,.\, x^{-1} \sin rx = 2\int_0^\infty dx \,.\, x^{-1} \sin rx = \pi,$$

observing that $\Gamma(1-n) = 1$ in limit $(n=0)$.

It *may* also be deduced from $\int_0^\infty \frac{\sin ax}{x \cos bx}$ by putting $b=0$, but I could not trust to the convergency of this integral, from which I deduced the value of $\int_0^\infty \frac{dx \sin rx}{x}$ at first.

The following is the Note sent me by Professor Clerk-Maxwell, and to which I have already alluded. It will be seen that when the first approximation is reasonably good, how distinctly the limits of general error can be defined by the elegant method of the learned Professor, and of which the main outlines are traced in this Note he has kindly allowed me to publish:—

"The error forces are of the form

$$X = u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$$

$$= \omega_1(\omega_1 x + \omega_2 y + \omega_3 z) - x(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

with similar expressions for Y and Z.

The resultant F, of X, Y, and Z, is in the direction of the perpendicular from x, y, z on a line through the centre, whose direction cosines are proportional to ω_1, ω_2, ω_3, and it is equal to this perpendicular multiplied by $$\Omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}.$$

This is the elementary theory of circular motion and centrifugal force.

It is manifest that, since F passes through the axis of motion of the shell, it can do no work *in virtue of the assumed* rotatory motion whose components are ω_1, ω_2, ω_3.

Hence, if $u = \omega_2 z - \omega_3 y$, &c.; $Xu + Yv + Zw = 0$ for every element of the system, as well as $\iiint (Xu + Yv + Zw)\, dx\, dy\, dz = 0$.

Also $Zy - Yz = 0$ for every element, because F passes through the axis of Ω. But it by no means follows that $Xu + Yv + Zw = 0$ if u, v, w refer to the actual motion in which X, Y, Z have free play.

In a uniform liquid within a closed vessel, any system of forces having a potential will be in equilibrium.

Hence, if we assume

$$X' = X + \frac{dV}{dx},$$

$$Y' = Y + \frac{dV}{dy},$$

$$Z' = Z + \frac{dV}{dz},$$

the force $F'(X', Y', Z')$ will be equivalent to F so far as the fluid motion is concerned.

It will only affect the pressure, not the motion.

The force F is in each shell perpendicular to the axis of rotation of that shell. But it is easy to see that the fluid might be guided by means of spherical septa, and therefore we may have a central force equivalent to F.

Let us then find V, so that F′ may be a central force, that is, that

$$X' = \frac{x}{r} R', \quad Y' = \frac{y}{r} R', \quad Z' = \frac{z}{r} R'.$$

If

$$V = \tfrac{1}{2} r^2 \Omega^2 - \tfrac{1}{2} (\omega_1 x + \omega_2 y + \omega_3 z)^2,$$

then

$$R' = r^2 \Omega \frac{d\Omega}{dr} - (x\omega_1 + y\omega_2 + z\omega_3) \left(x \frac{d\omega_1}{dr} + y \frac{d\omega_2}{dr} + z \frac{d\omega_3}{dr} \right).$$

Here, then, we have a central force which will keep the motion right. It vanishes whenever there is no variation of Ω in magnitude or direction. All that it does is to prevent the different shells from running into one another, which they would do on account of their different angular velocities. I think that an evaluation of R, or its maximum value, would give a limit of error in the solution."

The following presents to the Library were received during the Recess:—

"A Sketch in the Theory of Screws," by R. Stawell Ball, F.R.S. (from "Hermathena," No. ii., 1874).

"Screw Coordinates and their applications to Problems in the Dynamics of a Rigid Body," by the same (Transactions of Royal Irish Academy, January 12th, 1874).

"Crelle," 77 Band, 4es Heft; 78 Band, 1es Heft.

"Journal of Institute of Actuaries," No. 95, April 1874, and No. 96, July 1874.

"Bulletin des Sciences Mathématiques," July, August, 1874.

"Monatsbericht," April, May, June, July, August, 1874.

"Annuaire de l'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique," 1874.

"Bulletins de l'Académie Royale . . . . de Belgique," 42^{me} année, 2^{me} série, tomes xxxv., xxxvi., 1873.

"Annali di Matematica," tom. vi., fasc. 2° and 3° (Maggio and Sett. 1874).

Specimen pages of a Table of the Logarithms of all Numbers up to One Million, in preparation by Edward Sang, F.R.S.E., shortened to nine figures from original calculations to fifteen places of decimals.

"Annual Report of the Board of Regents of the Smithsonian Institution, Washington," 1873.

"Inhaltsverzeichniss der Abhandlungen der K. Akad. der Wissenschaften zu Berlin aus den Jahren 1822 bis 1872 nach den Klassen geordnet," Berlin, 1873.

"Jahrbuch über die Fortschritte der Mathematik," Vierter Band, Jahrgang 1872, Heft 1.

"Association Française pour l'avancement des Sciences, Congrès de Lyon, 1873," Paris. (Contains three papers by M. Mannheim.)

"Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades, von Axel Harnack" (from "Sitzungsberichte der physikalisch-medicinischen Societät zu Erlangen," 13 Juli, 1874).

"Nuove Richerche di Geometria pura sulle cubiche gobbe ed in ispecie sulla parabola gobba," par L. Cremona (Istituto di Bologna, 26 Nov. 1863, tomo iii., serie 2^{a}): from the Author to Dr. Hirst, by whom it was presented to the Society.

"Mémoires présentés à l'Académie des Sciences (tome xxii., No. 12) sur les surfaces trajectoires des points d'une figure de forme invariable dont le déplacement est assujetti à quatre conditions," par M. Mannheim: presented by Mr. Spottiswoode.

"Ueber eine Klasse binärer Formen:" from Prof. Klein ("Sitzungsberichte . . . zu Erlangen," 13 Juli, 1874).

"Washington Observations for 1872. Appendix II.—Report on the difference of Longitude between Washington and Detroit, Michigan; Carlin, Nevada; and Austin, Nevada;" by J. R. Eastman, Professor of Mathematics, U.S.N. Washington, 1874.

"Washington Observations for 1870. Appendix on the R. A. of the Equatorial Fundamental Stars and the Corrections necessary to reduce the R. A. of different Catalogues to a mean homogeneous system, by Simon Newcomb; prepared at the U.S.N. Observatory, by order of Rear-Admiral B. F. Sands, U.S.N."

"Bulletin de la Société Mathématique de France, publié par les Sécrétaires," tome ii., September, No. 4, Paris.

"Researches in the Dynamics of a Rigid Body by the aid of the Theory of Screws," by R. Stawell Ball, F.R.S., LL.D. (Phil. Trans., read June 19th, 1873).

"Sur la Fonction Exponentielle," par M. Hermite: presented by the Author.

APPENDIX.

The conversion of circular into general rectilinear motion (p. 4) is considered at length in a Friday Evening's Discourse at the Royal Institution (January 23rd, 1874). The title of Mr. Sylvester's address is, "On recent Discoveries in Mechanical Conversion of Motion." The paper, with a postscript, occupies twenty octavo pages, and gives a historical sketch of the subject.

An abstract of Mr. Marsham Adams' two communications will give an idea of the uses to which he applies the Mensurator and Cœlometer (p. 5). We may here state that the latter instrument has been adopted at the Royal Naval College, Greenwich, in the Examination Department of the Board of Trade, and in the training ships "Conway" and "Worcester."

The Mensurator is an instrument designed primarily for the instantaneous solution of triangles, but capable, from its construction, of many other uses, such as illustrating most of Euclid's theorems with regard to the triangle, of performing Addition, Subtraction, Rule of Three, and extraction of Square Root—of solving Quadratics and Simple Binomial Equations, and of reducing to mechanism some part

of Analytical Geometry. Many of these processes, however, are so simple as to be far more suitable for educational purposes, or for a general audience, than for a Society like the present; and I shall therefore only touch upon the solution of the two kinds of equations, and upon the representation of the tangent to an ellipse. Perhaps, however, I may be permitted to say that if any member should desire to go more thoroughly into the subject, a copy of my handbook will be at his service.

[Mr. Adams then applied the instrument to the mechanical solution of the following equations:—

$$x^2 + px = q^2, \qquad x^2 + px = -q^2,$$

$$\left.\begin{aligned} \frac{x}{m_1} + \frac{y}{n_1} &= 1 \\ \frac{x}{m_2} + \frac{y}{n_2} &= 1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{x}{30} + \frac{y}{32} &= 1 \\ \frac{x}{45} + \frac{y}{24} &= 1 \end{aligned}\right\}.]$$

Analytical Geometry.

Since the instrument becomes rigid only when three conditions are given, it follows that when two only are given, the vertex of the tri-. angle moves, and thus illustrates the conception of a locus. Thus, for example, if the base and altitude of the triangle be given, the vertex moves in a straight line parallel to the base.

Now let us call the sum of base and level the Departure; and the foot of the altitude the Plumb. Then, the altitude and departure form the co-ordinates of the moving vertex, and will be connected by those relations which are expressed in the ordinary equations to the curve An instance of this has been exhibited in the solution of simultaneous equations of x and y, by means of the intersection of the lines represented by the respective equations.

Again, if we set the base and move both the vertex along the traverse and the plumb along the axis simultaneously in a given ratio, we shall have a conic described by the vertex, which will be an Ellipse, Parabola, or Hyperbola, according as the traverse is given less than, equal to, or greater than the departure. And the given ratio of the traverse to the departure will be the eccentricity of the curve. The pivot will be the focus of the curve. The position of the vertex of the triangle when the deflection is 0, will be the vertex of the curve. The length of the altitude when the deflection is 90°, will be the Semi Latus Rectum. The position of the plumb when its distance from the vertex of the curve is equal to the corresponding traverse, will be the centre of the curve. The length of that traverse will be the Semi Axis

Major; and the length of the corresponding altitude the Semi Axis Minor.

If x_0 and y_0 be intercepts made by tangents at point $x'y'$ in ellipse,

$$\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1.$$

Then $$x_0x' = m^2,\ y_0y' = n^2.$$

In order, therefore, to find the point where the tangent cuts axis of x, we set the Plumb (*i.e.*, the foot of the Altitude) to the centre of ellipse obtained as above, and the extending of base to the vertex.

Then make Altitude $= x'$.

And Screw Elevation.

Next, bring Altitude to m.

Then Departure will be x_0.

Screw Base, and make Level $= x'$, and Altitude y'. The Vector will be a tangent at the point required.

The Cœlometer is an instrument for illustrating the various ideas necessary for elementary Astronomy, just as the Mensurator illustrates those of elementary Geometry. It consists of three parts, viz., an ordinary Globe, a Sun frame, and a Star frame, the latter being removable at pleasure. The following are some of the conceptions and relations which can be illustrated by the Cœlometer :—

1. The Terrestrial Globe. 2. The Zenith of any place on the Earth's surface. 3. The Celestial Meridian of any place on the Earth's surface, divided into measures of ten degrees. 4. The Prime Vertical. 5. The True Horizon. 6. The Apparent Horizon. 7. The Correction for Dip. 8. The Amplitudes. 9. The Solstices. 10. The Equinoctial. 11. The Ecliptic divided into signs. 12. The same divided into months. 13. The Coincidence of the Signs and the Months, *i.e.*, the Calendar. 14. The first point of Aries. 15. The Equinox. 16. The Obliquity of the Ecliptic. 17. The Solstitial Colure. 18. The Seasons. 19. The Meaning of the Tropical Circles. 20. The Meaning of the Arctic Circles. 21. The Correction for Parallax and Refraction. 22. The Sidereal day divided into hours. 23. The Apparent Solar day at any place, at any time of year. 24. The Mean Solar day. 25. The Mean time at Greenwich corresponding to any given hour at a given place. 26. The Mean time at any place corresponding to any given hour at Greenwich. 27. The Hour angle. 28. The Apparent Daily Motion of the Sun relatively to the horizon of any given place. 29. The Real Daily Motion of the horizon of any given place relatively to the Sun. 30. The Apparent Annual Motion of the Sun relatively to the Earth. 31. The Real Annual Motion of the Earth relatively to the

Sun. 32. Right Ascension divided into hours. 33. Declination divided into degrees. 34. The Precession of the Equinoxes. 35. The effect of the Precession on the Right Ascension of the Stars. 36. The Epoch. 37. The hours of Sunrise and Sunset, at any place, on any day of the year. 38. Celestial Latitude and Longitude. 39. The True and Apparent Altitudes of heavenly bodies above the horizon of any place. 40. The principle upon which the Latitude of any place is found by observing the Meridian Altitude of the Sun. 41. The principle of the same when an ex-meridian observation is taken. 42. The principle of the same from observation of the Pole Star. 43. The principle upon which Greenwich Mean time is calculated by the Chronometer. 44. The principle upon which observations for Longitude are taken about 9 a.m. instead of Noon. 45. The principle upon which the Longitude is found from observation of the Moon. 46. The principal Stars and Constellations. 47. The position of the principal Stars visible at a given place, for a given day and hour throughout the year. 48. The determination of the time at any place on the Earth's Surface, by the culmination of a given Star. 49. The various days on which different stars are visible for any given place. 50. The Celestial Phenomena of a voyage to the Antipodes.

Mr. Perigal's note on "Link Trammels" (p. 25) put in a claim for Jopling as having anticipated Peaucellier. We give it here, though Mr. Perigal's views were opposed by Messrs. Sylvester and Roberts.

Link Trammels. — "The Lozenge-Trammel of Peaucellier is a modified reproduction of Jopling's 'Double Cranks' Trammel,* invented about 1822 (probably before Peaucellier was born). Peaucellier's links, being arranged as parallelograms, therefore more easily understood by mathematicians, fortunately have attracted the attention of Sylvester, who has recognised their capabilities, and I hope will develope very many of the resultant curves, &c.

"I propose to call the curves *Caténites*, and the instruments *Catenegraphs*. The latter name may be objected to as hybrid; but as Jopling was English, and Peaucellier is French, their joint production may be considered hybrid also; so perhaps the term may be allowable under the extenuating circumstances that the golden egg was laid by Jopling, impregnated by Peaucellier, hatched and reared by Sylvester."

We fear that Professor Clerk Maxwell's paper "On Hamilton's

* J. Jopling's Double Cranks, description of, 4to, London, 1825. (L. $\frac{4}{18}$ Society of Arts Catalogue.)

Characteristic Function for a Narrow Beam of Light" (p. 25) is lost; at any rate the referee has mislaid it, and cannot now lay his hands upon it.

We may remark that, at the February Meeting, Messrs. Herbert Spencer and G. H. Lewes were present as visitors. Subsequently having learned the addresses of the Members of the Society from the Secretary, Mr. Spencer sent to all or nearly all the names on the Members' List a copy of two pamphlets: "Mr. Herbert Spencer and the British Quarterly Review;" "Appendices to a late pamphlet entitled, &c." The former is a reply to criticisms, a portion of which appeared in the "Fortnightly Review;" the latter contains letters and parts of letters which appeared in "Nature," between March 26th and June 18th, 1874; the correspondents being Mr. Spencer on the one side, and Messrs. J. F. Moulton and R. B. Hayward on the other. The discussion was mainly concerned with Newton's Laws of Motion, particularly the second Law.

We here remind Members that, in addition to the Society's Proceedings, Mr. A. J. Ellis, F.R.S., has sent to each Member a copy of his "Algebra Identified with Geometry." (p. 38.)

The following correction has been received from Mr. Monro (see pp. 74—78):

There is an inaccuracy near the foot of p. 76, which may be corrected with sufficient perspicuity by beginning after the third logarithmic equation, as follows:—

$$\log \frac{U_n}{(m+1)u_n} = \frac{\lambda}{2}\left(\frac{1}{x} - \frac{1}{x'}\right)\left(1 + \frac{\lambda^2}{3}\left[\frac{1}{x} + \frac{1}{x'}\right]\right) - \ldots\ldots *$$

This is of the order $\frac{\lambda}{m}$ or $\frac{1}{\sqrt{m}}$ when λ is of the order $\sqrt{m}$, and the series may be written $a_1\frac{\lambda}{m} + a_2\frac{\lambda^2}{m^2} + \ldots\ldots$,

where it is enough to say that a_1 and a_2 are of the order m^0 and do not change sign with λ. Now form a double term by taking into account equal and opposite values of λ: and here it will be convenient to replace the suffix n by a suffix λ, so that

$$\frac{U_\lambda}{m+1} = u_\lambda \log^{-1}\left(a_1\frac{\lambda}{m} + a_2\frac{\lambda^2}{m^2} + \ldots\right),$$

$$= u_\lambda\left(1 + a_1\frac{\lambda}{m} + \left[a_2 + \frac{a_1^2}{2}\right]\frac{\lambda^2}{m^2} + \ldots\right),$$

$$\frac{U_{-\lambda}}{m+1} = u_{-\lambda}\left(1 - a_1\frac{\lambda}{m} + \left[a_2 + \frac{a_1^2}{2}\right]\frac{\lambda^2}{m^2} + \ldots\right);\ \text{and then}$$

$$\frac{U_\lambda + U_{-\lambda}}{m+1} = (u_\lambda + u_{-\lambda})\left(1 + \left[a_2 + \frac{a_1^2}{2}\right]\frac{\lambda^2}{m^2} + \ldots\right) + (u_\lambda - u_{-\lambda})\left(a_1\frac{\lambda}{m} + \ldots\right).$$

* Here, and in the first logarithmic equation, the sign of the first term neglected is properly −, not +.

It is a well-known consequence of Laplace's investigation of the deductive theorem, that $u_\lambda - u_{-\lambda}$ is of an order lower by $\frac{1}{\sqrt{m}}$ than $u_\lambda + u_{-\lambda}$. Therefore we only neglect terms of an order lower by $\frac{1}{m}$ than that which we preserve when we write

$$U_\lambda + U_{-\lambda} = m\,(u_\lambda + u_{-\lambda}).$$

For integration we have $d\varpi = \frac{d\lambda}{m}$; and the limits corresponding to $\frac{n \pm \frac{1}{2}}{m}$ are $\lambda \pm \frac{1}{2}$ for both terms, u_λ and $u_{-\lambda}$. Therefore the approximation, commenced in the last line but one of the page, is correct for the term u_λ, and a similar one will be correct for the term $u_{-\lambda}$, with a result differing only in the sign of λ: so that we have

$$\frac{m^2\lambda^2}{2xx'}\left(\frac{u_\lambda}{(x+\lambda)(x'-\lambda)} + \frac{u_{-\lambda}}{(x-\lambda)(x'+\lambda)}\right),$$

which, if we neglect $\frac{1}{m}$ again in comparison with unity, becomes $\left(\frac{m\lambda}{xx'}\right)^2 u_\lambda + u_{-\lambda}$, for a superior limit of the error incurred by taking

$$\int_{\lambda - \frac{1}{2}}^{\lambda + \frac{1}{2}} (U_\lambda + U_{-\lambda})\,\frac{d\lambda}{m} = u_\lambda + u_{-\lambda}.$$

The irregular order in which papers appear in the present volume of Proceedings is very much to be regretted, but it could hardly have been otherwise unless there had been a greater delay * than there has been in the appearance of the several numbers. It is expected that this irregularity will not occur in the future, as the crisis in the Society's affairs, which was to some extent the cause of it, has passed, and owing to Lord Rayleigh's munificent gift (p. 89) is hardly likely to arise again.

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* The delay in many cases arose from causes over which the Secretaries had no control.

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed in extenso, or an abstract of it given in the Proceedings.*]

ADAMS, W. MARSHAM, B.A.

*The Mensurator and Cælometer, pp. 141—144.

CAYLEY, A., F.R.S.

*On Steiner's Surface, 14—25.

*On certain Constructions for Bicircular Quartics, 29—31.

*A Geometrical Interpretation of the Equations obtained by Equating to Zero the Resultant and the Discriminants of Two Binary Quantics, 31—33.

CLIFFORD, W. K., F.R.S.

*Graphic Representation of the Harmonic Components of a Periodic Motion, 11—14.

The Foundations of the Differential Calculus, and of Dynamics, 25, 27.

Free Motion of a Solid in Elliptic Space, 28.

CROFTON, M. W., F.R.S.

*A Method of treating the Kinematical Question of the most general displacement of a Solid in Space, 25—27.

DARWIN, G. H., M.A.

On Probable Error in Statistics, with a Note on the Combination of Statistics, 38.

ELLIS, A. J., F.R.S.

Algebra considered as the Calculus of Similar Triangles, 38. (Brought out in the form of an octavo pamphlet of 84 pp. A copy was presented to every Member of the London Mathematical Society.)

GLAISHER, J. W. L., M.A.

*On the Transformation of Continued Products into Continued Fractions, 78—88.

GRIFFITHS, J., M.A.

*On the Cartesian Equation of the Circle which cuts Three given Circles at given Angles, 33—35.

*On a Remarkable Relation between the Difference of two Fagnanian Arcs of an Ellipse of eccentricity e, and that of two corresponding Arcs of a Hyperbola of eccentricity $\frac{1}{e}$, 95, 96.

HIRST, T. A., F.R.S.

*On the Correlation of Two Planes, 40—70.

MAXWELL, J. CLERK, F.R.S.

On Hamilton's Characteristic Function for a Narrow Beam of Light, 25, 144. (It is feared that this Paper has been inadvertently lost by one of the Referees.)

Merrifield, C. W., F.R.S.
*Determination of the Form of the Dome of Uniform Stress, 113—119.

Monro, C. J., B.A.
*Note on the Inversion of Bernoulli's Theorem in Probabilities, 74—78, 145, 146.

Perigal, H., F.R.A.S.
*Link Trammels, 144.

Rayleigh, Lord, F.R.S.
*Vibrations of Membranes, 9, 10.
*Note on the Numerical Calculation of the Roots of Fluctuating Functions, 119—124.

Roberts, Samuel, M.A.
*Note on the Expression of the Length of the Arc of a Cartesian by Elliptic Functions, 6—9.
*On the Parallel Surfaces of Developables and Curves of Double Curvature, 90—94.

Röhrs, J. H., M.A.
*Spherical and Cylindric Motion in Viscous Fluid, 125—139.

Routh, E. J., F.R.S.
*Stability of a Dynamical System with Two Independent Motions, 97—99.
*On Rocking Stones, 100.
*Small Oscillations to any Degree of Approximation, 101—105.

Smith, H. J. S., F.R.S.
On the Higher Singularities of Plane Curves, 39.

Spottiswoode, W., F.R.S.
*On the Contact of Quadrics with other Surfaces, 70—74.

Sylvester, J. J., F.R.S.
Conversion of Circular into General Rectilinear Motion, and into Motion in Conics and Higher Plane Curves, 4, 141.

Taylor, H. M., M.A.
*Inversion, with Special Reference to the Inversion of an Anchor Ring or Torus, 105—112.

Wolstenholme, J., M.A.
*On another System of Poristic Equations, 35—37.

GENERAL INDEX.

Accounts, Auditor of, 4, 9.
Ampère, 116.
Anchor-ring, axis, axial plane, transverse plane, 107.
Annali di Matematica, 64.
Area, 12.
Bertrand, 79.
Bessel, 120.
Bunyakovski, 77.
Cartes de Visite, 3.
Casey, 112.
Caténites, Catenegraphs, 144.
Cayley, 34, 96, 112, 123.
Chasles, 40, 42, 45.
Circle, Bifid, 29.
Clebsch, 63, 66.
Complex, 61.
Connex, 63.
——— Conjugate, 63.
Cournot, 77.

Cranks, Double, 144.
Crelle, 79.
De Morgan, 77, 122.
Des Cartes, 98.
Educational Times and "Reprints," 66.
Encke, 123.
Euler, 88.
Fourier, 11, 131.
Fries, 77.
Galloway, 77.
Hansteen, Prof., death of, 4.
Hayward, 12, 145.
Helmholtz, 125, 131.
Institution, Royal, Friday's Discourse of, 141.
Jacobi, 7.
Jopling, 144.
Journals subscribed for, 3.
Laplace, 74—77.
Library, presents to, 3, 5, 6, 11, 25, 28, 29, 39, 40, 89.
Lines, Conjugate, 40.
Mathematics, Quarterly Journal of, 112.
Maxwell, 112, 125.
Members, Number of, 1.
Messenger of Mathematics, 84.
Moulton, 145.
Muir, 88.
Obituary, 1.
Ovals, Cartesian, 46, &c.
Papers read, Summary of, 1—3.
Peaucellier, 144.
Philosophical Magazine, 78, 88.
Plücker, 92.
Points, Conjugate, 40.
Poisson, 130.
Pratt, 119.
Precession and Nutation, 125, 127, 131.
Quadric Inversion, 64.
Quarterly Review, British, 145.
Rankine, 113.
Rayleigh's, Lord, gift, 89.
Report, Secretaries, 1—4.
Reye, 64.
Roberts, M., 7, 8.
Schröter, 44.
Societies, Proceedings exchanged with, 3.
Society, Royal, Proceedings of, 64.
Spencer, Herbert, 145.
Spiric, 30.
Stern, 88.
Stokes, 120, 125, 131, 132, 135.
Sturm, 50, 56.
Sylvester, 78, 81—88, 141—144.
Thomson, Sir W., 125.
Todhunter, 74, 75, 77, 119.
Venturoli, 119.
Wallis, 78, 79, 87.

BIBL.
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1956-1